Mathematics 700 Homework.

1. This problem is a warm up for some of the machinery that we will need to understand normal forms for similarity which is the high point of the class. Let V be a vector space and let U, W be subspaces of V so that $U + W = V$ and $U \cap W = \{0\}$. Then you have shown in an old homework (page 40, problem 9) that every $v \in V$ can then be uniquely expressed as $v = u + w$ where $u \in U$ and $w \in W$. In this case we write $V = U \oplus W$ and say that V is the ***direct sum*** of U and W. (To be redundant (but explicit) if from now on you hear that V is a direct sum of U and W (or see the formula $V = U \oplus W$) then you are immediately to think that U and W are subspaces of V with $U + W = V$, $U \cap W = \{0\}$ and also remember that in particular this means that every $v \in V$ can be uniquely written as $v = u + w$ with $u \in U$ and $w \in W$.)

   (a) If $V = U \oplus W$ and $u_1, \ldots, u_k$ is a basis of U and $w_1, \ldots, w_m$ is a basis of W then show $u_1, \ldots, u_k, w_1, \ldots, w_m$ is a basis of V. Thus

   $$\dim V = \dim U + \dim W.$$

   REMARK: This also follows from the formula $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$.

   (b) If $V = U \oplus W$ then define a linear map $P : V \to V$ by

   $$Pv = u \quad \text{where } v = u + w \text{ with } u \in U \text{ and } w \in W$$

   P is called the ***projection of*** V ***onto*** U ***with kernel*** W.

   i. Show $\operatorname{Image} P = U$ and $\operatorname{Ker} P = W$.
   ii. Show that $P^2 = P$.
   iii. Show $Pu = u$ if and only if $u \in U$.

   (c) Let $P : V \to V$ be a linear map that satisfies $P^2 = P$. (REMARK: Linear maps P satisfying $P^2 = P$ are called ***projections***. They occur very naturally in both algebra and analysis.)

   i. Show

   $$V = \operatorname{Image} P \oplus \operatorname{Ker} P.$$

   (HINT: Every $v \in V$ can be written as $v = Pv + (v - Pv)$.)

   ii. Let $k = \operatorname{rank} P = \dim \operatorname{Image} P$ and let $v_1, \ldots, v_k$ be a basis for $\operatorname{Image} P$ and $v_{k+1}, \ldots, v_n$ (where $n = \dim V$) be a basis of $\operatorname{Ker} P$. Then by the first problem above $v_1, \ldots, v_n$ is a basis of V. What is the matrix of P in this basis?

   (d) Let $T : V \to V$ be a linear map that satisfies the equation $T^2 = I$ (where I is the identity transformation.) Then as a variant on the last problem

   i. Show

   $$V = \operatorname{Ker}(T - I) \oplus \operatorname{Ker}(T + I).$$

   ii. If $v_1, \ldots, v_n$ is a basis if V so that $v_1, \ldots, v_k$ is a basis of $\operatorname{Ker}(T - I)$ and $v_{k+1}, \ldots, v_n$ is a basis of $\operatorname{Ker}(T + I)$ then what is the matrix of T in this basis?
   iii. Can you generalize to T that satisfy other polynomial equations? For example what if T satisfies $T^3 = T$? **Remark:** If you don't see a generalization don't spend much time on this. If you do see what is going on it is worth pursuing as it will give you a leg up on much of what we will be doing.

Quotient Spaces

Let V be a vector space over the field $\mathbf{F}$ and W a subspace of V. Then define an equivalence relation $\sim_W$ by

$$v_1 \sim_W v_2 \quad \text{if and only if} \quad v_2 - v_1 \in W.$$

Problem 1 Show that this is an equivalence relation.

Denote by $[v]_W$ the equivalence class of $v \in V$ under the equivalence relation $\sim_W$. That is

$$[v]_W := \{u \in V : u \sim_W v\}.$$

Problem 2 Show $[v]_W = v + W$ where $v + W = \{v + w : w \in W\}$.

Let V/W be the set of all equivalence classes of $\sim_W$. That is

$$V/W := \{[v]_W : v \in V\} = \{v + W : v \in V\}.$$

The equivalence class $[v]_W = v + W$ is often called the ***coset of*** v ***in*** V/W.

Problem 3 Let $V = \mathbf{R}^2$ and let W be the subspace of points of V of points (x, y) with $y = -2x$. Then draw pictures of the coset of $(1, 1)$ in V/W and the coset of $(3, -2)$ in V/W. What is a geometric description of the coset of $v \in \mathbf{R}$ in V/W?

Define a sum and scalar multiplication in V/W by

$$[v_1]_W + [v_2]_W := [v_1 + v_2]_W \quad c[v]_W := [cv]_W$$

where $v_1, v_2, v \in V$ and $c \in \mathbf{F}$.

Problem 4 Show this is well defined. The term ***well defined*** is used in mathematics to mean "is independent of the choices made in the definition". In this particular case this means you need to show

$$[v_1]_W = [v_1']_W \quad \text{and} \quad [v_2]_W = [v_2']_W \qquad \text{implies} \qquad [v_1 + v_2]_W = [v_1' + v_2']_W$$

and

$$[v]_W = [v']_W \qquad \text{implies} \qquad [cv]_W = [cv']_W.$$

Proposition 0.1 *With these operations V/W is a vector space.*

Problem 5 Prove this.

Problem 6 If V is finite dimensional then what is the dimension of V/W in terms of $\dim V$ and $\dim W$? Prove your answer is correct.

Problem 7 In the example of problem 3 draw some pictures of cosets $v_2 + W$ and $v_2 + W$ what their sum $(v_1 + W) + (v_2 + W)$ and the linear combination $2(v_1 + W) - 3(v_2 + W)$ for a few choices of v_1 and v_2.

Other Problems

Note that what I have been calling ***eigenvalues*** is what the book calls ***characteristic values***.

1. Let
$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
be a two by two matrix. As usual $\operatorname{trace} A = a + d$, $\det A = ad - bc$. Set $f(x) = x^2 - (\operatorname{trace} A)x + \det A$. Show $f(A) = 0$. Show the minimal polynomial of A divides $f(x)$.

2. Let A be the matrix

$$A := \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

   (a) Use the last problem to find the minimal polynomial of A.
   (b) Find the eigenvalues of A.
   (c) Find a formula for A^n. HINT: Find a basis of $\mathbf{R}^2$ of eigenvectors of A. In this basis it is easy to compute powers of A. Do the computation in this basis and then change back to the standard basis.

3. Define a sequence $x_0 = 0$, $x_1 = 1$ and $x_{k+1} = 3x_k - 2x_{k-1}$ for $k \geq 1$. Let A be the matrix in the last problem. Then show

$$A \begin{bmatrix} x_{k-1} \\ x_k \end{bmatrix} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \qquad \text{and} \qquad A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}.$$

   Now find a formula for x_k. HINT: By the last problem you have a forumla for A^k.