Mathematics 700 Homework

1. One of the applied mathematicians here has expressed very strong opinions about people teaching Math 700 and letting students get by without knowing some basic facts about the range and null spaces of matrices. While he has not gone so far as to say that if he catches any of you not knowing these facts that he will hunt me down and break my kneecaps with a baseball bat, I am going to insure an ambulatory future for myself by making sure you do know both of facts in question. Let A be an $m \times n$ matrix over the field $\mathbf{F}$. Then as usual we can view A as a linear map $A : \mathbf{F}^n \to \mathbf{F}^n$ by matrix multiplication $X \mapsto AX$ and where $\mathbf{F}^m$ and $\mathbf{F}^n$ are spaces of column vectors. Then explain why the following are true.

   (a) The null space of A is the space of vectors of orthogonal to all the rows of A. (We have not given an official definition of what it means for a row vector to be orthogonal to a column vector over the field $\mathbf{F}$ so you should include such a definition.)

   (b) The range of A is the span of the columns of A.

2. The rank plus nullity theorem is one of the great (and underrated) existence theorems in elementary mathematics. Here are some examples for you to work out.

   (a) (From algebra.) It is a basic fact that if $p(x)$ is a polynomial of degree $\leq n$ over the field $\mathbf{F}$ that has $(n+1)$ roots then $p(x)$ is the zero polynomial. Use this fact to show that for any elements $z_0, \ldots, z_n, w_0, \ldots w_n \in \mathbf{F}$ with $z_0, \ldots, z_n$ distinct there is a unique polynomial $p(x)$ of degree $\leq n$ so that $p(z_i) = w_i$ for $i = 1, \ldots, n$. This is often stated loosely and a little imprecisely as: It is possible to assign the values of a polynomial $\leq n$ at $n+1$ points arbitrarily. HINT: Let $\mathcal{P}_n$ be the polynomials of degree $\leq n$ and the linear linear map $V : \mathcal{P}_n \to \mathbf{F}^{n+1}$ by $V(p) := (p(z_0), \ldots, p(z_n))$ and apply the rank plus nullity theorem.

   (b) (From ordinary differential equations.) Let $C^2(\mathbf{R})$ be the vector space of all real valued functions on $\mathbf{R}$ that are twice continuously differentiable. Let a and b be real numbers and let V be the two dimensional subspaces of $C^2(\mathbf{R})$ spanned by the two functions $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$. Let a_0, a_1, and a_2 be be any real numbers and define $L : C^2(\mathbf{R}) \to C^2(\mathbf{R})$ by $Ly := a_2 y'' + a_1 y' + a_0 y$. Assume that the only solution to $Ly = 0$ in V is $y = 0$. Then show that for any $f \in V$ there is a unique $y_p \in V$ so that $Ly_p = f$. COMMENT: This is basically a justification for a special case of the method of undetermined coefficients you have seen in your differential equations class. It in not much harder to justify the entire method alongs these same lines.

   (c) (From partial differential equations.) Let $\mathcal{HP}_n^2$ be the homogeneous polynomials of degree n in the two variables x and y with real coefficients. That is elements of $\mathcal{HP}_n^2$ are of the form $a_0 x^n + a_1 x^{n-1} y + \cdots + a_k x^{n-k} y^k + \cdots + a_n y^n$ where $a_0, \ldots, a_n$ are real numbers. A ***harmonic polynomial of degree*** n is an element h of

$\mathcal{HP}^1_n$ that satisfies the partial differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Show that there are lots of harmonic polynomials. HINT: Let $\mathcal{H}^2_n$ be the space of all harmonic polynomials of degree n. Define a linear map $\Delta \mathcal{HP}^2_n \to \mathcal{HP}^2_{n-2}$ by

$$\Delta := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Now what are the dimensions of $\mathcal{HP}^2_n$ and $\mathcal{HP}^2_{n-2}$ and what is the null space of Δ? With just a little work you should be able to give the exact dimension of $\mathcal{H}^2_n$.

(d) (From algebraic geometry) We say that a curve in the plane $\mathbf{R}^n$ has a ***polynomial parameterization*** iff it is of the form

$$c(t) = (A(t), B(t)) = (a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m, b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n)$$

where $A(t)$ and $B(t)$ are polynomials as indicated. Then show that every curve with an polynomial parameterization is algebraic in the sense that there is a nonzero polynomial $p(x, y)$ in two variables so that

$$p(A(t), B(t)) \equiv 0.$$

HINT: Let $\mathcal{P}^2_k$ be the vector space of polynomials of degree total degree $\leq k$ in the two variables x and y. Let $\mathcal{P}^1_d$ be the polynomials of degree $\leq d$ in the variable t. Given the two polynomials $A(t)$ and $B(t)$ assume that $m \leq n$ and define a linear map $E : \mathcal{P}^2_k \to \mathcal{P}^1_{nk}$ by

$$E(f)(t) := f(A(t), B(t)).$$

Now compute dimensions of $\mathcal{P}^1_{nk}$ and $\mathcal{P}^2_k$ and let k get large. Are you able to give a bound on the degree of p in terms of the dimensions of $A(t)$ and $B(t)$?