Mathematics 700 Test 1

1. (20 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear map
$$T(x, y) = (2x - 3y, x - 2y).$$
   (a) Find the matrix of T in the standard basis.
   (b) Find the matrix of T in the ordered basis $(0, 3)$, $(1, -1)$.
   (c) Find the inverse of T. (The answer should be expressed in the same form as T and not as a matrix.)
   (d) Find a basis for both the range and null space of the linear transformation $T - I$.
   (e) Find T^2.
   (f) Find T^{1995}

2. (10 points) Give an example of linear maps $T : \mathbf{R}^2 \to \mathbf{R}^4$ and $S : \mathbf{R}^4 \to \mathbf{R}^2$ so that ST is invertible.

3. (10 points) Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ and $S : \mathbf{R}^2 \to \mathbf{R}^4$ be linear maps. Show $\text{nullity}(ST) \geq 2$.

4. (10 points) If u, v, w are linearly independent vectors in the vector space V show that u, $u + v$ and $u + v + w$ are also linearly independent.

5. (10 points) Let $v_1 = (1, 0, 0)$, $v_2 = (2, 1, 0)$, $v_3 = (3, 2, 1)$. Then v_1, v_2, v_3 is a basis of $\mathbf{R}^3$. Find the basis of $\mathbf{R}^{3*}$ dual to v_1, v_2, v_3.

6. (10 points) Let U and W be subspaces of the finite dimensional vector space V. Assume that $\dim U + \dim W = \dim V$ and $U + W = V$. Then show $V = U \oplus W$.

7. (10 points) Let $\mathbf{C} = \{x + iy : x, y \in \mathbf{R}\}$ be the complex numbers. Then $\mathbf{C}$ is a two dimensional vector space over the real numbers $\mathbf{R}$. Let $T : \mathbf{C} \to \mathbf{C}$ be multiplication by $a + bi$, that is $Tz = (a + bi)z$ Then find the matrix of T in basis 1, i of $\mathbf{C}$.

8. (10 points) Find explicitly a linear map $S : \mathbf{R}^2 \to \mathbf{R}^3$ so that $S(2, 3) = (1, 2, 3)$ and $S(1, 2) = (4, 5, 6)$.

9. (10 points) Gives examples of
   (a) Matrices A and B with $AB \neq BA$.
   (b) A matrix with $A^3 = 0$ but $A^2 \neq 0$.

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This is to be done in three hours in one setting. I would prefer that it is closed book, but if you feel that you have to look up a something write me a note like "I looked up the definition of rank and used it in problems numbers 2 and 5". I will then take a little off on these problems, say 20%. This is due in class next Tuesday. Good luck.