Another Approach to Acsoli's Theorem.

This set of notes and problems is to show how Ascoli's theorem can be deduced from the Tychonoff product theorem. I will not try for the most generality, but that the method generalizes should be clear. We start with some elementary topology.

Proposition 1. *If K is a compact subset of a Hausdorff space X, then K is a closed subset of X.*

Problem 1. Prove this. $\square$

Proposition 2. *It X and Y are topological spaces and $f\colon X \to Y$ continuous, then if $K \subseteq X$ is compact, then $f[K]$ is a compact subset of Y.*

Problem 2. Prove this. $\square$

The following is elementary but quite useful.

Theorem 3. *Let X and Y be topological spaces with X compact and Y Hausdorff and $f\colon X \to Y$ a continuous bijection, then f is a homeomorphism.*

Problem 3. Prove this. Hint: As f is a bijection the inverse $f^{-1}\colon Y \to X$ exists. Showing the f^{-1} is continuous is equivalent to showing that f is open. Let $U \subset X$ be open. Then $K := X \setminus U$ is closed in X and therefore compact. Thus $f[K]$ is compact in Y and therefore $f[U] = Y \setminus f[K]$ is open. $\square$

Here is an example of use of this theorem. Let $f\colon [a,b] \to [c,d]$ be a continuous strictly increasing function with $f(a) = c$ and $f(b) = d$. Then the inverse $f^{-1}\colon [c,d] \to [a,b]$ is continuous. This is because $[a,b]$ is compact and $[c,d]$ is Hausdorff.

Exercise 1. Try proving this directly without use Theorem 3. $\square$

Corollary 4. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Assume that $\mathcal{T}_2$ is compact and $\mathcal{T}_1$ is Hausdorff. Then $\mathcal{T}_1 = \mathcal{T}_2$.*

Problem 4. Prove this. Hint: Apply Theorem 3 to the identity map $I\colon (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$. $\square$

For the rest of these notes X and Y are both compact metric space. Let $C > 0$ be a positive constant and let

$$\mathcal{L} := \{f\colon X \to Y : d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2)\}.$$

That is $\mathcal{L}$ is the set of all Lipschitz maps from X to Y with Lipschitz constant C. Let $C(X,Y)$ be the metric space of all continuous function $f\colon X \to Y$ with the metric

$$d_C(f,g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Thus convergence with respect to the metric d_C is just uniform convergence. Our goal is to show that $\mathcal{L}$ is a compact subset of $C(X, Y)$.

We consider the space Y^X of all functions $f\colon X \to Y$ with the product topology. As Y is compact this Tychonoff's theorem implies that Y^X us a compact Hausdorff space.

Lemma 5. *Show that $\mathcal{L}$ is a closed, and thus compact, subset of Y^X.*

Problem 5. Prove this. □

Lemma 6. *Let $\mathcal{T}_C$ be the topology on $\mathcal{L}$ induced by the metric d_C and let $\mathcal{T}_P$ be the topology induced on $\mathcal{L}$ by the product topology on Y^X. Then $\mathcal{T}_C \subseteq \mathcal{T}_P$.*

Problem 6. Prove this. HINT: Let $U \in \mathcal{T}_C$. Then it is required to show that U is open in the $\mathcal{T}_P$ topology. Let $f \in U$, then, by the definition of the metric topology, there is an $r > 0$ such that $B_C(f, r) := \{g \in \mathcal{L} : d_C(f, g) < r\} \subseteq U$. As X is compact there is a finite set $\{x_1, \ldots, x_n\} \subset X$ such that for every point $x \in X$ there an x_i with $d_X(x, x_i) < r/(3C)$. Let $V := \{g \in \mathcal{L} : d_Y(f(x_i), g(x_i) < r/3, \text{ for } i = 1, \ldots, n\}$. This is open in the $\mathcal{T}_P$ topology. If $g \in V$, then for any $x \in X$ choose an x_i such that $d_X(x, x_i) < r/(3C)$. Then

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), g(x_i)) + d_Y(g(x_i), f(x_i)) \\ &< C d_X(x, x_i) + \frac{r}{3} + C d_X(x, x_i) \\ &< C\frac{r}{3C} + \frac{r}{3} + C\frac{r}{3C} = r. \end{aligned}$$

This holds for all x, so $d_C(f, g) < r$ and therefore $g \in B_C(f, r)$. As g was an arbitrary element of V, this implies $f \in V \subset B_C(f, r) \subseteq U$. Therefore U contains a $\mathcal{T}_P$ neighborhood, V, about any of its points, f, and so $U \in \mathcal{T}_P$. □

Theorem 7. *With the topology induced by the metric d_C, the set $\mathcal{L}$ is a compact subset of $C(X, Y)$.*

Problem 7. Prove this. HINT: By Lemma 5 the topology $\mathcal{T}_P$ is compact. The topology T_C is Hausdorff and by Lemma 6 the inclustion $\mathcal{T}_C \subseteq \mathcal{T}_P$ holds. By Corollary 4 this implies $\mathcal{T}_C = \mathcal{T}_P$. □

Remark 8. It only takes minor variants on this argument to prove the full form of Ascoli's Theorem.