The Vitali Covering Theorem.

Let $(\mathbf{X}, d)$ be a metric space such that closed bounded sets are compact and let μ^* a nonzero outer measure on $\mathbf{X}$ such that all Borel subsets of $\mathbf{X}$ are measurable with respect to μ^*. The restriction of μ^* to the measurable subsets of $\mathbf{X}$ will be denoted by μ. We assume that μ satisfies a doubling condition

$$\mu B(x, 2r) \leq C \mu B(x, r).$$

Let E be a subset of $\mathbf{X}$ then a collection $\mathcal{V}$ of subset I of $\mathbf{X}$ is a ***Vitali cover*** of E iff

(1) Each $I \in \mathcal{V}$ is a closed set,
(2) For all $x \in E$ and $\varepsilon > 0$ there is an $I \in \mathcal{V}$ such with $x \in I$ and $r(I) < \varepsilon$, and
(3) There is an $\alpha > 0$ such for each $I \in \mathcal{V}$ there is a ball $B(x_I, r_I)$ with $I \subset B(x_I, R_I)$ and

$$\mu I \geq \alpha \mu B(x_I, r_I).$$

If $\mathcal{V}$ is a Vitali cover of some set let

$$r(\mathcal{V}) = \sup_{I \in \mathcal{V}} r_I.$$

Theorem 1. *Let $U \subset \mathbf{X}$ be a bounded open set with $\mu U < \infty$ and let $E \subseteq U$ be any set (not necessarily measurable). Let $\mathcal{V}$ be Vitali cover of E such that $I \subset U$ for all $I \in \mathcal{V}$. Let $\rho \in (0, 1)$ and let a sequence $\{I_k\}_{k=1}^{\infty}$ of elements of $\mathcal{V}$ be chosen so that I_1 is any element of $\mathcal{V}$ such that*

$$r_{I_1} \geq \rho r(\mathcal{V}).$$

If $I_1, \ldots, I_k$ have been chosen let

$$\mathcal{V}_k := \{I \in \mathcal{V} : I \cap (I_1 \cup \cdots \cup I_k) = \varnothing, \text{ and } E \cap I \neq \varnothing\}$$

and let I_{k+1} be any element of $\mathcal{V}_k$ such that

$$r_{I_{k+1}} \geq \rho r(\mathcal{V}_k).$$

Then

$$\mu^* \left(E \setminus \bigcup_{k=1}^{\infty} I_k \right) = 0.$$

Here are some problems that outline a proof. To simplify notation let $r_k = r_{I_k}$ and $x_k = x_{I_k}$.

Problem 1. $\lim_{k \to \infty} r_k = 0$. Hint: As the sets $\{I_k\}_{k=1}^{\infty}$ are pairwise disjoint we have

$$\mu U \geq \mu \left(\bigcup_{k=1}^{\infty} I_k \right) = \sum_{k=1}^{\infty} \mu I_k.$$

Therefore $\sum_{k=1}^{\infty} \mu I_k < \infty$ which implies that

$$0 = \lim_{k \to \infty} \mu I_k \geq \lim_{k \to \infty} \alpha \mu B(x_k, r_k) \geq 0.$$

Thus $\lim_{k \to \infty} \mu B(x_k, r_k) = 0$. Now assume, toward a contradiction, that $\lim_{k \to \infty} r_k \neq 0$. Then there is a $\delta > 0$ and a subsequence r_{k_ℓ} such that $r_{k_\ell} \geq 2\delta$ for all ℓ and, by local complactness, by going to a subsequence we can assume that $\lim_{\ell \to \infty} x_{k_\ell} = x_*$ exists. Then for all sufficiently large ℓ we have that $B(x_*, \delta) \subset B(x_{k_\ell}, r_\ell)$. Use this to show that $\mu B(x_*, \delta) = 0$ and then use the doubling condition to show that this implies $\mu\mathbf{X} = 0$, which contradicts that μ is not the zero measure. $\square$

Problem 2. Let $x \in E \setminus \bigcup_{j=1}^{k} I_j$. Show that $x \in \bigcup_{j=k+1}^{\infty} B(x_j, (2\rho^{-1}+1)r_j)$. Hint: The set $\bigcup_{j=1}^{k} I_j$ is closed and $\mathcal{V}$ is a Vitali cover thus there is an $I \in \mathcal{V}$ with $x \in I$ and $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{k} I_j\right) = \varnothing$. Let $n \geq k$ and assume that $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{n} I_j\right) = \varnothing$. Show

$$r_{n+1} \geq \rho r(\mathcal{V}_n) \geq \rho r_I$$

and this implies that $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{n+1} I_j\right) \neq \varnothing$ for some $n \geq k$ (as $r_{n+1} \to 0$). Let n be the smallest integer where $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{n+1} I_j\right) \neq \varnothing$. Then explain why $r_{n+1} \geq \rho r(\mathcal{V}_n) \geq \rho r_I$ and

$$B(x_I, r_I) \cap B(x_{n+1}, r_{n+1}) \neq \varnothing.$$

Show these facts and that $x \in B(x_I, r_I)$ implies that $x \in B(x_{n+1}, (2\rho^{-1} + 1)r_{n+1})$ which completes the proof. $\square$

Problem 3. Complete the proof of Theorem 1. Hint: Let $F := E \setminus \bigcup_{k=1}^{\infty} I_k$. We wish to show that $\mu^* F = 0$. By the last problem we have for each postive integer n that $F \subset \bigcup_{k=n+1}^{\infty} B(x_k, (2\rho^{-1} + 1)r_k)$. Therefore by sub-additivity

$$\mu^* F \subset \sum_{k=n+1}^{\infty} \mu B(x_k, (2\rho^{-1} + 1)r_k).$$

Whence if the series

$$\sum_{k=1}^{\infty} \mu B(x_k, (2\rho^{-1} + 1)r_k)$$

converges we are done. Explain why it does converge.

Problem 4. Let U be a bounded open set in the plane. For any such set let

$$r(U) = \sup\{r : \ U \text{ contians a disk of radius } r.\}.$$

Define a sequence of closed disks as follows. D_1 is any closed disk in U with radius at least $\frac{1}{2}r(U)$. If $D_1, \ldots, D_n$ have been defined let $U_n := U \setminus (D_1 \cup \cdots \cup D_n)$ and D_{n+1} be any disk contained in U_n with radius at least $\frac{1}{2}r(U_n)$. Show that

$$\sum_{k=1}^{\infty} \operatorname{Area}(D_k) = \operatorname{Area}(U).$$