Mathematics 242 Final, Take Home Portion Show your work to get credit. This is due at the beginning of the final, 5:30pm Monday, December 6.

(1) (20 points) We have two tanks, tank A and tank B, each holding 100 gallons and two pumps one that pumps water from tank A to tank B at 5 gallons/hour, and another that pumps water from tank B to tank A at the same rate of 5 gallons/hour. When the pubs are started tank A has 100 gallons of a mixture of salt and water with 10 pounds of salt in the solution. Tank B starts with 100 gallons of pure water. (See figure 1.)

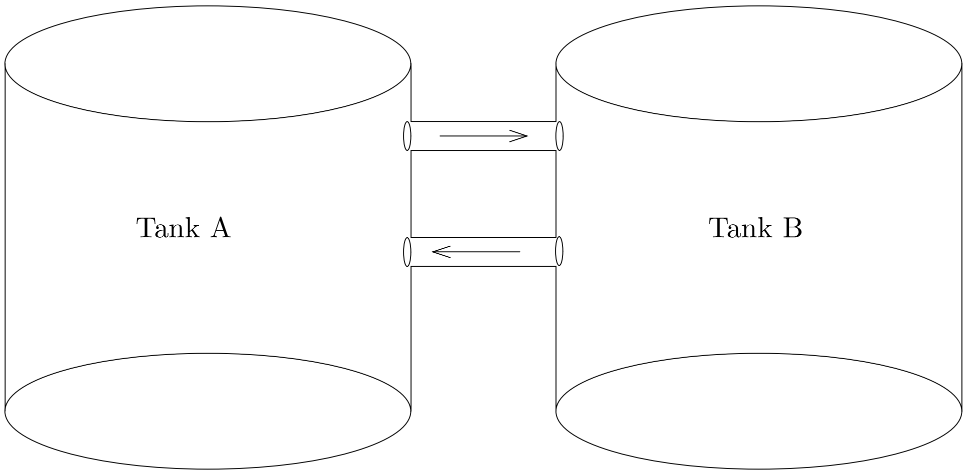

FIGURE 1. Each tank contains 100 gallons and well mixed solution is pumped between them at a rate of 5 gallons/hour. When the pumps are started tank A has 100 gallons of a water/salt solution with 10 lbs of salt and tank B has pure water.

(a) Let $x(t)$ be the amount, in pounds, of salt in tank A t hours after the pumps are turned on and $y(t)$ the number of pounds of salt in tank B t hours after the pumps are turned on. Explain why $x(t)$ and $y(t)$ satisfy the initial value problem

$$\begin{aligned} x'(t) &= -\frac{5}{100}x(t) + \frac{5}{100}y(t) & \qquad x(0) &= 10 \\ y'(t) &= \frac{5}{100}x(t) - \frac{5}{100}y(t) & \qquad y(0) &= 0 \end{aligned}$$

(b) Solve for $x(t)$ and $y(t)$.

(2) (15 points) Find an initial value problem satisfied by the function $f(t) = te^t \cos(2t)$, and use it to find the Laplace transform $\mathcal{L}\{f(t)\}$.

(3) (15 points) Find the solution to the initial value problem

$$x'(t) + x(t) = \begin{cases} 1, & t < 1; \\ 2t - 1 & 1 \le t. \end{cases}$$

and $x(0) = 2$. HINT: Note that the right hand side of the above can be written as

$$1 + u(t-1)(2t-2) = \begin{cases} 1, & t < 1; \\ 2t - 1 & 1 \le t. \end{cases}$$

Mathematics 242 Test #3, Take Home Portion

(a) (25 points) Solve

$$x'(t) + 5x(t) = \begin{cases} 0, & t < 2; \\ 12e^{3t}, & 2 \le t. \end{cases}$$

and $x(0) = 6$

Solution: The initial value problem can be rewritten as

$$x'(t) + 5x(t) = 12u(t - s)e^{3t}, \qquad x(0) = 6.$$

Let $X(s) = \mathcal{L}\{x(t)\}$. Then taking Laplace transforms gives and using $\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s) - 6$.

$$sX(s) - 6 + 5X(s) = \mathcal{L}\{12u(t-2)e^{3t}\} = \mathcal{L}\{12u(t-2)e^{3(t-2)+6}\} = 12e^6 \frac{e^{-2s}}{s-3}$$

where we have used the formula

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}.$$

Solving for $X(s)$ gives

$$X(s) = \frac{6}{s+5} + e^6 e^{-2s}\left(\frac{12}{(s-3)(s+5)}\right) = \frac{6}{s+5} + \frac{3}{2}e^6 e^{-2s}\left(\frac{1}{s-3} - \frac{1}{s+5}\right)$$

Taking inverse Laplace transforms gives

$$\boxed{x(t) = 6e^{-5t} + \frac{3e^6}{2}u(t-2)\left(e^{3(t-2)} - e^{-5(t-2)}\right).}$$

(b) (20 points) Let $x(t)$ and $y(t)$ be related by

$$\begin{aligned} x'(t) &= 2x(t) - 3y(t) \\ y'(t) &= -3x(t) + 2y(t) \end{aligned}$$

and

$$x(0) = 4, \quad y(0) = -6.$$

Take the Laplace transform of these equation to get a system of algebraic equations for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$. Solve these equations for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and then take the inverse Laplace transforms to find $x(t)$ and $y(t)$.

Solution: Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$. Using $\mathcal{L}\{x(t)\} = sX(s) - x(0) = sX(s) - 4$ and $\mathcal{L}\{y(t)\} = sY(s) - x(0) = sY(s) + 6$ we get

$$\begin{aligned} sX(s) - 4 &= 2X(s) - 3Y(s) \\ sY(s) + 6 &= -3X(s) + sY(s) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} (s-2)X(s) + 3Y(s) &= 4 \\ 3X(s) + (s-2)Y(s) &= -6 \end{aligned}$$

These can be solved by Crammer's rule to give

$$\begin{aligned} X(s) &= \frac{4s+10}{(s+1)(s-5)} = \frac{5}{s-5} - \frac{1}{s+1}, \\ Y(s) &= \frac{-6s}{(s+1)(s-5)} = \frac{-5}{s-5} - \frac{1}{s+1}. \end{aligned}$$

Taking inverse Laplace transforms then gives

$$\boxed{x(t) = 5e^{5t} - e^{-t}, \qquad y(t) = -5e^{5t} - e^{-t}.}$$

Here is a mistake that was make on the last test that you should not make again. The Laplace transform of a product is not the product of the Laplace transforms. That is

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Here is an example. Let $f(t) = e^{at}$ and $g(t) = e^{bt}$. Then

$$\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{e^{at}e^{bt}\} = \mathcal{L}\{e^{(a+b)t} = \frac{1}{s-(a+b)}$$

and

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{e^{at}\}\mathcal{L}\{e^{bt}\} = \frac{1}{s-a}\,\frac{1}{s-b} = \frac{1}{(s-a)(s-b)}.$$

The two are clearly not equal.