Math 554, Test 1, Take Home Portion.

This will be due in class at the beginning of class on Monday, September 30. You are not to talk to each other about these problems.

Definition 1. A subset, S, of the real numbers, $\mathbb{R}$, is ***compact*** iff every open cover, $\mathcal{H}$, of S has a finite sub-cover.

With this terminology the Heine-Borel Theorem can be stated as

Theorem 2. *Every closed bounded subset of* $\mathbb{R}$ *is compact.*

We wish to show the converse of this, that is that every compact subset of $\mathbb{R}$ is closed and bounded. We split this into two parts.

Problem 1 (15 points). If S is compact, then S is bounded. *Hint:* Let S be compact and let $\mathcal{H} = \{(-n, n) : n = 1, 2, 3, \ldots\}$.

(a) Show $\mathcal{H}$ is an open cover of S. (This should involving using the Archimedean Principle.)
(b) Use that $\mathcal{H}$ has a finite sub-cover to show S is bounded.

Problem 2 (15 points). If S is compact, then S is closed. *Hint:* Use a proof by contradiction. Assume that S is compact, but not closed. Then S has a limit point x_0 with $x_0 \notin S$. For each positive integer n let

$$U_n = (-\infty, x_0 - 1/n) \cup (x_0 + 1/n, \infty)$$

and let

$$\mathcal{H} = \{U_n : n = 1, 2, 3, \ldots\}$$

(a) Show that $\mathcal{H}$ is an open cover of S. (This will use that $x_0 \notin S$ and the Archimedean Principle).
(b) Use that $\mathcal{H}$ has a finite sub-cover and that x_0 is a limit point to get a contradiction.

And here is a limit problem.

Problem 3 (15 points). Let f and g be defined in a deleted neighborhood of x_0. Assume that f is bounded, as $|f(x)| \leq A$ for all x in the domain of f and that $\lim_{x \to x_0} g(x) = 0$. Show

$$\lim_{x \to x_0} f(x)g(x) = 0.$$

Remarks: Recall proofs about limits should start with "Let $\varepsilon > 0$". For a limit proof to be correct that you have to say how δ is chosen. Finally I don't want to see your scratch work.