Math 554 Homework

Now that we know definitions of things like open set, closed set, interior point and such like we wish to understand these better by proving basic results about them. Most of what follows here either follows by writing down the definitions and using just a bit of logic, or can be found in the text.

Proposition 1. *The intersection of a finite number of open sets is an open set.*

Problem 1. Prove this. *Hint:* Let $U_1, \ldots, U_n$ be the open sets. Let $x_0 \in U_1 \cap \cdots U_n$. As each U_j there is an $\varepsilon_j > 0$ so that $(x_0 - \varepsilon_j, x_0 + \varepsilon_j) \subseteq U_j$ and consider $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$.

Proposition 2. *Let $\{U_\alpha\}_{\alpha \in A}$ be a collection, possibly infinite, of open sets. Then the union*

$$U = \bigcup_{\alpha \in A} U_\alpha$$

is also open.

Problem 2. Prove this.

Problem 3. Give an example of an infinite collection of open sets whose intersection is the closed interval $[0, 1]$. Thus the intersection of infinitely many open sets need not be open.

Dual to these results about open sets, there are results about closed set.

Proposition 3. *The finite union of closed sets is closed.*

Problem 4. Prove this. *Hint:* $(F_1 \cup \cdots \cup F_n)^c = F_1^c \cap \cdots \cap F_n^c$.

Proposition 4. *Let $\{F_\alpha\}_{\alpha \in A}$ be a collection, possibly infinite, of closed sets. Then the intersection*

$$F = \bigcap_{\alpha \in A} F_\alpha$$

is also closed.

Problem 5. Prove this.

Problem 6. Give an example of a infinite union of closed sets that is not closed.

Here is what many texts use as the definition of the closure:

Proposition 5. *The closure, $\overline{S}$, of S is the set of points x_0 so that every neighborhood of x_0 contains a point of S.*

Problem 7. Prove this.

Proposition 6. *The closure, $\overline{S}$, of as set S is a closed set.*

Problem 8. Prove this.

Proposition 7. *The boundary, ∂S, of any set S is closed.*

Problem 9. Prove this.

Proposition 8. *For any set S, the closure, $\overline{S}$, of S is the union of S with the limit points of S.*

Problem 10. Prove this.

Problem 11. Problem 11 on page 28 of the text.