Math 554 Homework

You should know the statements of the **Heine-Borel Theorem** (Page 25 of the text) and the (Bolzano-Weierstrass Theorem) (Page 27).

Theorem 1 (Bolzano-Weierstrass)**.** *Every bounded infinite set of real numbers has a limit point.*

Problem 1. Prove this along the following lines. Let S be a bounded infinite subset of $\mathbb{R}$. As S is bounded we can assume that $S \subseteq [a, b]$ for some closed interval $[a, b]$. Let

$$F := \{x : (x, \infty) \cap S \text{ is finite}\}.$$

(a) Show that if $x \in F$ and $y \geq x$, then $y \in F$.
(b) Show $F \neq \varnothing$ (because $b \in F$) and that a is a lower bound for F. Thus $\beta = \inf F$ exists.
(c) Show β is a limit point of S. *Hint:* Towards a contradiction assume that β is not a limit point of H. Then β has a neighborhood $(\beta - \varepsilon, \beta + \varepsilon)$ that does not contain any point of S other than possibly β. Let $\beta_1 \in (\beta, \beta + \varepsilon)$, then $S \cap (\beta_1, \infty)$ is finite (as β is a lower bound for F.) Explain why $S \cap (\beta - \varepsilon, \infty)$ is finite, and why this gives a contradiction.

Problem 2. In the text to problems 18, 20 a, b, c, d on page 28.