Math 554 Homework

Here is a warm up.

Proposition 1. *If f is defined in a deleted neighborhood of x_0 and*

$$\lim_{x \to x_0} f(x) = L$$

then for any constant c

$$\lim_{x \to x_0} cf(x) = cL$$

Problem 1. Prove this. *Hint:* $|cf(x) - cL| = |c||f(x) - L|$ and so if $|f(x) - L| < \varepsilon/|c|$, then $|cf(x) - cL| < \varepsilon$. You may have to do a separate argument for $c = 0$, but that case is easy.

Proposition 2. *If f and g are defined on a deleted neighborhood of x_0 and*

$$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M$$

then for any constants c_1 and c_2

$$\lim_{x \to x_0} (c_1 f(x) + c_2 g(x)) = c_1 L + c_2 M.$$

Problem 2. Prove this. *Hint:* The scratch would start with $|c_1 f(x) + c_2 g(x) - (c_1 L + c_2 M)| \le |c_1||f(x) - L| + |c_2||g(x) - M|$. Now make each of these terms less than $\varepsilon/2$.

We now get to the rather more tricky case of products. We start with a special case

Proposition 3. *Let f and g be defined in deleted neighborhoods of x_0 and*

$$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M.$$

Let us make the extra assumption that f is bounded, say

$$|f(x)| \le A$$

for some constant $A > 0$. Then

$$\lim_{x \to x_0} f(x)g(x) = LM.$$

Problem 3. Prove this. *Hint:* Here is a start on the scratch work. This uses the adding and subtracting trick.

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&\le |f(x)||g(x) - M| + |f(x) - L|M && \text{(triangle inequality)} \\
&\le A|g(x) - M| + |f(x) - L||M| && \text{(as } |f(x)| \le A\text{)}
\end{aligned}$$

So if $|f(x) - L| < \varepsilon/(2|M|)$ and $|g(x) - M| < \varepsilon/(2A)$ we are done.

Proposition 4. *If f is defined in a deleted neighborhood of x_0 and*

$$\lim_{x \to x_0} f(x) = L$$

then $f(x)$ is bounded near x_0. More precisely there is a δ so that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x)| < |L| + 1.$$

Problem 4. Prove this. *Hint:* Use $\varepsilon = 1$ in the definition of a limit, the adding and subtracting trick in the form, $|f(x)| = |f(x) - L + L|$, and then the triangle inequality.

Remark. Because $f(x) = \frac{1}{x}$ is not bounded in any neighborhood of x, Proposition 4 shows that $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Theorem 5. *If f and g are defined in a deleted neighborhood of x_0 and*

$$\lim_{x \to x_0} f(x) = L, \qquad \lim_{x \to x_0} g(x) = M,$$

then

$$\lim_{x \to x_0} f(x)g(x) = LM.$$

Problem 5. Prove this. *Hint:* By Proposition 4 there is a $\delta_1 > 0$ so that

$$0 < |x - x_0| < \delta_1 \qquad \Longrightarrow \qquad |f(x)| \leq |L| + 1.$$

Now you should be in a position to use the same argument as in the proof of Proposition 3 with $A = |L| + 1$.