# Math 554 Homework.

The last main topic to cover on derivatives is the derivatives of inverse functions.

**Theorem 1.** *Let $f\colon (a,b) \to (A,B)$ be continuous and bijective. Assume that $f$ is differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$. Then $f^{-1}\colon (A,B)$ is differentiable at $y_0 := f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}.$$

*Proof.* For $y \in (A,B)$ let $x = f^{-1}(y)$, then we also have $y = f(x)$. A previous theorem tells us that $f^{-1}$ is continuous. Therefore

$$\lim_{y\to y_0} x = \lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0.$$

and

$$\lim_{x\to x_0} y = \lim_{x\to x_0} f(x) = f(x_0) = y_0.$$

That is $y \to y_0$ if and only if $x \to x_0$. Thus

$$\begin{aligned}
(f^{-1})'(y_0) &= \lim_{y\to y_0} \frac{f^{-1}(y) - f^{-1}(x_0)}{y - y_0} \\
&= \lim_{y\to y_0} \frac{x - x_0}{f(x) - f(x_0)} && \text{(Substitute } x = f^{-1}(y) \text{ etc.)} \\
&= \lim_{x\to x_0} \frac{x - x_0}{f(x) - f(x_0)} && \text{(As } y \to y_0 \iff x \to x_0.\text{)} \\
&= \lim_{x\to x_0} \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}} && \text{(A bit of algebra.)} \\
&= \frac{1}{\displaystyle\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}} && \left(\text{As } h(t) = \frac{1}{t} \text{ is continuous.}\right) \\
&= \frac{1}{f'(x_0)} && \text{(As } f'(x_0) \text{ exists.)}
\end{aligned}$$

This completes the proof. □

*Example* 2. We know that the function $f(x) = x^2$ is differentiable on $(0,\infty)$ with $f'(x) \neq 0$ on this interval. Thus the last theorem tells us that the inverse $g(x) = \sqrt{x}$ is also differentiable on $(0,\infty)$. To find its derivative note that

$$g(x)^2 = x.$$

Take the derivative of this to get

$$2g(x)g'(x) = 1.$$

That is

$$g'(x) = \frac{1}{2g(x)} = \frac{1}{2\sqrt{x}}.$$

**Problem** 1. Let $f(x) = x^n$ where $n \neq 0$ is an integer (positive or negative). Then $f$ is differentiable on $(0, \infty)$ with $f'(x) = nx^{n-1}$ so that $f'(x) \neq 0$ on $(0, \infty)$. Let $g(x) = x^{1/n}$ be the inverse of $f$ on $(0, \infty)$. The last theorem implies $g$ is differentiable on $(0, \infty)$. As $g$ is the inverse of $f$ we have

$$g(x)^n = x.$$

Take the derivative of this to derive a formula for $g'(x)$. *Hint:* The answer is $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$.

**Problem** 2. Let $m, n$ be non-zero integers and let $r = \frac{m}{n}$. Set $f(x) = x^r$. Show $f$ is differentiable on $(0, \infty)$ and

$$f'(x) = rx^{r-1}.$$

*Hint:* Write $f(x) = \left(x^{1/n}\right)^m$ and combine rules that we know.

**Problem** 3. Let $f$ be defined and differentiable on $(0, \infty)$ and assume that $f(x) > 0$ on $(0, \infty)$ and that $f'(x) = cf(x)$ where $c \neq 0$ is a constant. Let $g$ be the inverse of $f(x)$. Find a formula for $g'(x)$. *Hint:* Start with $f(g(x)) = x$ and take the derivative.

**Problem** 4. To generalize the last problem, let $f\colon (a, b) \to (A, B)$ be differentiable and have inverse $g\colon (A, B) \to (a, b)$. Assume that there is a function $\phi$ such that

$$f'(x) = \phi(f(x)).$$

Then show

$$g'(x) = \frac{1}{\phi(x)}.$$

*Remark* 3. As an example of the last problem let $f(x) = \sin(x)$ on $(-\pi/2, \pi/2)$. Then $f\colon (-\pi/2, \pi/2) \to (-1, 1)$ is differentiable with inverse $g(x) = \arcsin(x)$. On $(-\pi/2, \pi/2)$ we have that $f'(x) = \cos(x)$ is positive and therefore

$$f'(x) = \cos(x) = \sqrt{1 - \sin^2(x)} = \sqrt{1 - f(x)^2} = \phi(f(x))$$

with $\phi(x) = \sqrt{1 - x^2}$. Whence

$$\arcsin'(x) = g'(x) = \frac{1}{\phi(x)} = \frac{1}{\sqrt{1 - x^2}}.$$