Math 554 Review for Test 1.

Definitions and Statements of Theorems. There will be definitions and statemenes of theorems on the test. Know the following statements and definitions.

(1) The basic properties of the real numbers as given by **(A)** – **(H)** on Pages 1–2 of the text. (You don't have to know which letter, i.e. **(C)**, corresponds to which property, you just need to know the properties.)
(2) b is an ***upper bound*** for the set S.
(3) a is a ***lower bound*** for the set S.
(4) β is the ***supremum*** of the set S, written $\beta = \sup S$.
(5) α is the ***infimum*** of the set S, written $\alpha = \inf S$.
(6) The ***completeness axiom*** Page 4 **(I)**.
(7) The ***Archimedean property***.
(8) D is ***dense*** in $\mathbb{R}$.
(9) The ***extended real number system*** and its rules of addition and multiplication. (Pages 7–9).
(10) A ε-***neighborhood*** of x_0.
(11) An ***open*** set.
(12) A ***closed*** set.
(13) A ***limit point*** of a set.
(14) The ***boundary*** of a set.
(15) The ***closure*** of a set.
(16) An ***isolated*** point of a set.
(17) An ***interior*** point of a set.
(18) An ***exterior*** point of a set.
(19) The ***Bolzano-Weierstrass Theorem***.
(20) $\mathcal{H}$ is an ***open cover*** of S.
(21) The ***Heine-Borel Theorem***.
(22) The definition of $\lim_{x\to x_0} f(x) = L$.

Here are some proofs you should know down cold.

- The proof of the Archimedean Principle from the completeness axiom.
- If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$ then $\lim_{x\to x_0}(f(x) - g(x)) = L - M$. This includes being able to deal with variants such as $\lim_{x\to x_0}(3f(x) - 2g(x)) = 3L - 2M$.
- The union of a finite number of open set is open.

Here are some sample questions grouped by subject. We started out by talking about the basic algebraic properties of the real numbers, and this included basic inequalities. We also used induction to prove some inequalities. Here are some basic problems related to this.

Problem 1. • Show $n! > 3^n$ for $n \geq 7$ (you can use that $7! = 5040 > 2187 = 3^7$). *Hint:* Use induction.

- Show $|x-1|<2$ implies $|x+3|<6$.
- Show $x \geq 10$ implies $\frac{1}{x} \leq \frac{1}{10}$.
- If for all $x>0$ the inequality $x^2 + \dfrac{1}{x^2} \geq 2$ holds. *Hint:* Rewrite this as $y^2 \geq 0$ for some expression y.
- If $a<b<0$, then $0<b^2<a^2$.
- Know the statement of the binomial theorem.

We next went into the basic properties of open and closed sets. Here are some problems related to that

Problem 2.
- Show that for any subset $S \subseteq \mathbb{R}$, that $\partial S = \partial S^c$.
- Show that $\overline{S \cup T} = \overline{S} \cup \overline{T}$.
- Give an example where $\partial(S \cap T) \neq \partial S \cap \partial T$.
- What is the set of limit points of the set $\{1/n : x = 1, 2, 3, \ldots\}$?
- Find an example of infinitely open sets whose intersection is closed.
- If D is a closed dense set, then $D = \mathbb{R}$. *Hint:* If D is dense what is its set of limit points.

We next looked at the completeness axiom and its consequences the Bolzano-Weierstrass Theorem and the Heine-Borel Theorem.

Here are problems related to these.

Problem 3.
- Show that if S is closed and bounded above, then $\sup S \in S$.
- (A variant of the last question.) Show that if S is bounded above, then $\sup S \in \partial S$.
- Give an example of a set S that is bounded above such that $\sup S \notin S$.
- Show that if S is bounded above, then $\partial S \neq \varnothing$. *Hint:* Consider $\beta = \sup S$ and show it is a boundary point.

Finally there are limits. As there already a fair amount material above, there will not be much on limits. Here are problems that would show you know the definition.

Problem 4.
- If $\lim_{x \to x_0} f(x) = L$, then show there is a $\delta > 0$ so that $0 < |x - x_0| < \delta$ implies $|f(x)| < |L| + .01$.
- Show $\lim_{x \to 5}(3x - 9) = 6$.
- Show $\lim_{x \to 2} x^2 = 4$.
- If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0}(2f(x) + 3g(x)) = 2L + 3M$.