Math 554, Final Name: Solutions.

The test is worth 150 points, but there are 165 points possible. Thus there are 15 points of extra credit available.

1. (a) State the definition of the limit

$$\lim_{x\to a} f(x) = L.$$

Solution: f is defined on a deleted neighborhood of a and for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon.$$

$\square$

(b) If

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

prove from the definition of limit that

$$\lim_{x\to a} (f(x) + g(x)) = L + M.$$

Solution: By the definition of the two limits existing there are $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} 0 < |x-a| < \delta_1 &\implies |f(x) - L| < \frac{\varepsilon}{2}, \\ 0 < |x-a| < \delta_2 &\implies |g(x) - M| < \frac{\varepsilon}{2}. \end{aligned}$$

Let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then $0 < |x-a| < \delta$ implies

$$\begin{aligned} |f(x) + g(x) - (L+M)| &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

which shows the limit is as desired. $\square$

2. (a) Define what it means for the set U to be open.

Solution: For every $x_0 \in U$ there is a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq U$. $\square$

Alternative Solution: Every point of U is an interior point of U. $\square$

(b) Define with it means for a function f to be continuous at x_0.

Solution: f is defined in a neighborhood of x_0 and for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - x_0| < \varepsilon.$$

$\square$

Alternative Solution: f is defined in a neighborhood of x_0 and $\lim_{x\to x_0} f(x) = f(x_0)$. $\square$

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Show the set $U = \{x : f(x) > 0\}$ is open.

Solution: Let $x_0 \in U$ and let $\varepsilon = f(x_0)$ in the definition of f being continuous at x_0. Then there is a $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

But then $f(x) > f(x_0) - \varepsilon = f(x_0) - f(x_0) = 0$. Thus we have that $x \in (x_0 - \delta, x_0 + \delta)$ implies $f(x) > 0$. Therefore $(x_0 - \delta, x_0 + \delta) \subseteq U$. As x_0 as any point of U this shows U is open. $\square$

3. (a) State what it means for a function to be ***differentiable*** at x_0.

Solution: f is differentiable at x_0 iff f is defined in a neighborhood of x_0 and the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In the case the limit exists its value, denoted by $f'(x_0)$, is the ***derivative*** of f at x_0 $\square$

(b) Show that if f is differentiable at x_0 then f is continuous at x_0.

Solution: As f is differentiable at x_0, it is defined in a neighborhood of x_0. We need to show that $\lim_{x \to x_0} f(x) = f(x_0)$. Using basic results about limits we have

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \left(f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right) \\
&= \lim_{x \to x_0} f(x_0) + \left(\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \left(\lim_{x \to x_0} (x - x_0) \right) \\
&= f(x_0) + f'(x_0) \cdot 0 \\
&= f(x_0).
\end{aligned}$$

$\square$

Alternative Solution: Or we can work directly with the ε-δ definition of the limit. As the derivative exists, f is defined in a neighborhood of x_0. Using $\varepsilon = 1$ in the definition of $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ we find a $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < 1.$$

But some algebra shows that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < 1 \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < (|f'(x_0)| + 1)\, |x - x_0|.$$

Let

$$\delta = \min \left\{ \delta_1, \frac{\varepsilon}{|f'(x_0)| + 1} \right\}.$$

Then if $|x - x_0| < \delta$

$$|f(x) - f(x_0)| \le (|f'(x_0)| + 1)\, |x - x_0| < (|f'(x_0)| + 1) \frac{\varepsilon}{|f'(x_0)| + 1} = \varepsilon$$

which is just what we wanted. $\square$

4. (a) State what it means for $\mathcal{U}$ to be an ***open cover*** of the set A.

Solution: $\mathcal{U}$ is an open cover of A iff $\mathcal{U}$ is a collection of open subsets of $\mathbb{R}$ and for each $a \in A$ there is a $U \in \mathcal{U}$ with $a \in U$. $\square$

(b) State the ***Heine-Borel Theorem***.

Solution: Let A be a closed bounded subset of $\mathbb{R}$. Then every open cover of A has a finite subset which is also a cover of A. $\square$

(c) Let f be defined on the closed bounded set A. Assume that for each $a \in A$ there is a $\delta_a > 0$ and a $C_a > 0$ such that

$$f(x) \geq C_a \quad \text{for all} \quad x \in (a - \delta_a, a + \delta_a).$$

Show there is a constant $B > 0$ such that $f(x) \geq B$ for all $x \in A$.

Solution: Let $\mathcal{U} = \{U_a : a \in A\}$ where $U_a = (a - \delta_a, a + \delta_a)$. Then each element of $\mathcal{U}$ is an open set and if $a \in A$, then $a \in U_a$ so $\mathcal{U}$ is an open cover of A. By the Heine-Borel Theorem there is a finite number of elements $a_1, a_2, \ldots, a_n \in A$ such that

$$A \subseteq U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}.$$

Let

$$B = \min\{C_{a_1}, C_{a_2}, \ldots, C_{a_n}\}.$$

Then $B = C_{a_k}$ for some k and thus $B = C_{a_k} > 0$ (this was an important point and only two people in the class pointed it out). If $x \in A$, then $x \in U_{a_j}$ for some j with $1 \leq j \leq n$ and therefore

$$f(x) \geq C_{a_j} \geq \min\{C_{a_1}, C_{a_2}, \ldots, C_{a_n}\} = B.$$

$\square$

5. (a) Define what it means for f to be ***uniformly continuous*** on all of $\mathbb{R}$.

Solution: f is uniformly continuous on $\mathbb{R}$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

(b) Show if f is uniformly continuous on $\mathbb{R}$, and $|f| \leq 3$ on $\mathbb{R}$ that f^2 is also uniformly on $\mathbb{R}$. *Hint:* $|f(x)^2 - f(y)^2| = |f(x) + f(y)||f(x) - f(y)|$.

Solution: Using the definition of uniform continuity we find a $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{6}.$$

Then if $x, y \in \mathbb{R}$ with $|x - y| < \delta$

$$\begin{aligned} |f(x)^2 - f(y)^2| &= |f(x) + f(y)||f(x) - f(y)| \\ &\le (|f(x)| + |f(y)|)\, |f(x) - f(y)| \\ &\le (3 + 3)|f(x) - f(y)| \\ &< 6\,\frac{\varepsilon}{6} \\ &= \varepsilon. \end{aligned}$$

□

6. (a) State the ***intermediate value theorem***.

Solution: If f is continuous on the closed interval $[a, b]$ and c is between $f(a)$ and $f(b)$, then there is a $x \in (a, b)$ with $f(x) = c$. □

(b) State the ***mean value theorem***.

Solution: If f is differentiable on (a, b) and continuous on $[a, b]$, then there is a $\xi \in (a, b)$ with

$$f(b) - f(a) = f'(\xi)(b - a).$$

□

(c) Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + \cos(x)$ is bijective (that is one to one and onto).

Proof that f is injective (one to one): As $f'(x) = 2 - \sin(x) \ge 2 - 1 = 1$ we have that f is strictly increasing. And a strictly increasing function is one to one. □

Alternate proof that f is injective: Let $x_1 \ne x_2$. Then by the mean value theorem there is a ξ between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \ne 0$$

as $f'(\xi) = 2 - \sin(\xi) \ge 2 - 1 = 1$. Thus if $x_1 \ne x_2$ it follows that $f(x_1) \ne f(x_2)$. Therefore f is injective. □

Proof that f is surjective (onto):

Let $y \in \mathbb{R}$ and let $a = \frac{y}{2} + 1$ and $b = \frac{y}{2} - 1$. Then

$$f(a) = 2\left(\frac{y}{2} + 1\right) + \cos(a) = y + 2 + \cos(a) \ge y + 2 - 1 = y + 1 > y$$

and

$$f(b) = 2\left(\frac{y}{2} - 1\right) + \cos(b) = y - 2 + \cos(b) \le y - 2 + 1 = y - 1 < y.$$

So by the intermediate value theorem there is a x between a and b with $f(x) = y$. As y was any element of $\mathbb{R}$, this shows that f is onto. □

(d) Is the inverse of f differentiable? Give a brief (two or three sentences) reason for your answer.

Solution: We have a theorem that tells that that if f is differentiable, bijective, and $f'(x) \ne 0$ for any x in the domain of f, then the inverse, f^{-1}, of f is differentiable. Thus f^{-1} is differentiable. □

7. (a) State the definition of $\liminf_{x\to a} f(x) = L$.

Solution:
$$\liminf_{x\to a} f(x) = \lim_{r\to 0^+} \big(\inf\{f(x) : 0 < |x-a| < r\}\big).$$

□

(b) Let $\liminf\limits_{x\to 3} f(x) = 5$. Show there is a $\delta > 0$ such that $f(x) > 4$ for $x \in (3-\delta, 3+\delta)$.

Solution: Let $\varepsilon = 1$. From the definition of $\liminf_{x\to 3} f(x) = 5$ there is a $\delta > 0$ such that
$$0 < r < \delta \qquad \Longrightarrow \qquad \big|\big(\inf\{f(x) : 0 < |x-a| < r\}\big) - 5\big| < \varepsilon.$$
That is
$$0 < r < \delta \qquad \Longrightarrow \qquad 5 - \varepsilon < \inf\{f(x) : 0 < |x-a| < r\} < 5 + \varepsilon.$$
Then if $x \in (3-\delta, 3+\delta)$ have that there is a r with $x < r < \delta$ and so
$$f(x) \ge \inf\{f(x) : 0 < |x-a| < r\} \ge 5 - \varepsilon = 4.$$

□

8. (a) Let f be 4 times differentiable on an open interval I. Give the ***third order Taylor expansion*** with Lagrange's form of the remainder (so that the remainder terms involves $f^{(4)}$) about the point $x_0 \in I$.

Solution: There is a ξ between x and x_0 such that
$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + f'''(x_0)\frac{(x-x_0)^3}{6} + f^{(4)}(\xi)\frac{(x-x_0)^4}{24}.$$

(b) Let f be 4 times differentiable on an open interval I and assume that $f^{(4)}(x) > 0$ for all $x \in I$. Assume there is a point $x_0 \in I$ such that
$$f'(x_0) = f''(x_0) = f'''(x_0) = 0.$$
Show that x_0 is a minimizer for f on I.

Solution: Using the Taylor expansion of the first part of the problem and using that the derivatives vanish we have that if $x \in I$ and $x \ne x_0$
$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + f'''(x_0)\frac{(x-x_0)^3}{6} + f^{(4)}(\xi)\frac{(x-x_0)^4}{24} \\ &= f(x_0) + f^{(4)}(\xi)\frac{(x-x_0)^4}{24} \\ &\ge f(x_0) + 0 \\ &= f(x_0) \end{aligned}$$
where we have used that $f^{(4)}(\xi) > 0$ and $(x-x_0)^4 > 0$. Thus $f(x) > f(x_0)$ for all $x \in I$ with $x \ne x_0$. Therefore x_0 is a minimizer of f on I. □

9. (a) Define what it means for the function f to be ***strictly convex*** on the interval I.

Solution: For any $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $x, y \in I$ the inequality
$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$
holds and equality holds in this inequality if and only if $x = y$.

Alternative Solution: For any $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and $x, y \in I$ with $x \neq y$ the inequality

$$f(\alpha x + \beta y) < \alpha f(x) + \beta f(y)$$

holds.

(b) State ***Jensen's inequality***.

Solution: Let f be convex on the interval I. Then for $x_1, \dots, x_n \in I$ and $\alpha_1, \dots, \alpha_n > 0$ with $\alpha_1 + \cdots + \alpha_n = 1$ the inequality

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n 0) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

holds. □

Alternative Solution: Let f be strictly convex on the interval I. Then for $x_1, \dots, x_n \in I$ and $\alpha_1, \dots, \alpha_n > 0$ with $\alpha_1 + \cdots | \alpha_n = 1$ the inequality

$$f(\alpha_1 x_1 + \cdots = \alpha_n x_n 0) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

holds, and equality holds in this inequality only when $x_1 = x_2 = \cdots = x_n$. □

(c) Show for positive numbers $x_1, x_2, \dots, x_n$ the inequality

$$(x_1 + x_2 + \cdots + x_n)^3 \leq n^2 \left(x_1^3 + x_2^3 + \cdots + x_n^3\right)$$

holds with equality if and only if $x_1 = x_1 = \cdots = x_n$. *Hint:* You may assume that $f(x) = x^3$ is strictly convex on $(0, \infty)$.

In the strictly convex version of Jensen's inequality let $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1/n$. Then

$$\begin{aligned} \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^3 &= f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \\ &\leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \\ &= \frac{x_1^3 + x_2^3 + \cdots + x_n^3}{n}. \end{aligned}$$

Multiply through by n^3 to get

$$(x_1 + x_2 + \cdots + x_n)^3 \leq n^2 \left(x_1^3 + x_2^3 + \cdots + x_n^3\right).$$

If equality holds in this inequality, then equality would hold in the strictly convex form of Jensen's inequality and we would have that $x_1 = x_2 = \cdots = x_n$. □