Review for Test 3

Here is an outline of some of the main topics we have covered since the last test.

(1) Properties of metric spaces.
   (a) Compactness. This was on the second test, but it is such a central topic in analysis that it will come up on this test also. In particular know that any sequence in a compact space has a convergent subsequence, and that any infinite subset of a compact space has a cluster point.
   (b) Connectedness. Know the definition of a metric space being connected (we had three equivalent definitions). Our big theorem here is that intervals in $\mathbf{R}$ are connected.
   (c) Some sample problems:
      (i) Let E be a metric space with $E = U \cup V$ with U and V nonempty open sets and $U \cap V = \varnothing$. Show that if $A \subseteq E$ is connected, then either $A \subseteq U$ or $A \subseteq V$.
      (ii) If $E = A \cup B$ with A and B nonempty and connected and $A \cap B \neq \varnothing$, then E is connected.
      (iii) If $A \subseteq \mathbf{R}$ and there are points a, b, c with $a < b < c$, $a, c \in A$ and $b \notin A$. Then A is not connected.
      (iv) Draw a picture of subsets A, B of $\mathbf{R}^2$ such that A and B are connected by $A \cap B$ is disconnected.
      (v) Draw a picture of subsets A, B of $\mathbf{R}^2$ such that A and B are both disconnected but $A \cup B$ is connected.

(2) Continuous functions between metric spaces.
   (a) You certainly need to know the ε–δ definition of continuity.
   (b) We have several equivalent ways to look at continuity:
      (i) A function between metric spaces is continuous if and only if preimages of open sets are open. *Sample problem:* Show that $\{(x, y) : y^2 > x^3 - 2x + 1\}$ is an open subset of $\mathbf{R}^2$.
      (ii) A function between metric spaces is continuous if and only if preimages of closed sets are closed. *Sample problem:* Let E be a metric space and let $p \in E$ and define $\rho_p \colon E \to \mathbf{R}$ by $\rho_p(x) := d(p, x)$ (that is $\rho_p(x)$ is the distance of x from p). First show that ρ_p is continuous on E by showing $|\rho_p(x) - \rho_p(y)| \leq d(x, y)$ and then use this to show that for any $p, q \in E$ the set $S = \{x : d(p, x) = d(q, x)\}$ is closed.
      (iii) A function between metric spaces is continuous if and only if it does the right thing to convergent sequences. See the Proposition on Page 74 of the text for the exact statement. *Sample problem:* If $f \colon \mathbf{R} \to \mathbf{R}$ is continuous, then the graph $G = \{(x, f(x)) : x \in \mathbf{R}\}$ is a closed subset of $\mathbf{R}^2$.

   (c) The composition of continuous functions is continuous. (The proposition on page 71 of the text.) You should be able to use prove this.

(3) Continuous functions on compact spaces.
   (a) The most basic result here is that the continuous image of a compact set is compact (be able to give a precise statement). *You should be able to prove this.* Recall that the idea of the proof is easy. Let $f\colon E \to E'$ be continuous with E compact. Let $\mathcal{V}$ be an open cover of $f[E]$. Then you show $\mathcal{V}^* := \{f^{-1}[U] : U \in \mathcal{V}\}$ is an open cover of E. This will have a finite subset $\{f^{-1}[U_1], \dots, f^{-1}[U_n]\}$ that covers E and you can then show that $\{U_1, \dots, U_n\}$ a cover of $f[E']$. *Sample problem:* Problem 16 on page 93 of the text. *Hint:* The $(f^{-1})^{-1} = f$ thus if C is a closed subset of E' we have $(f^{-1})^{-1}[C] = f[C]$ is the continuous image of a compact set. And we know that a compact subset of a metric space is closed and that a function is continuous if and only if the preimages of closed sets are closed.
   (b) Real valued functions on a compact space: Let $f\colon E \to \mathbf{R}$ be a continuous function where E is compact. Then f achieves its maximum and minimum. *Sample problem:* Show that if $f\colon [-1,1] \to \mathbf{R}$ is given by $f(x) = \frac{10x^2-10+\sqrt{5}x^3}{\sqrt{x^4+1}}$ that there is some $a \in [-1,1]$ such that $f(x) \le f(a)$ for all $x \in [-1,1]$.

(4) Continuous functions on connected spaces.
   (a) The most basic result here is that the continuous image of a connected space is connected. *You should both know an exact statement of this and how to prove it.*
   (b) (Intimidate Value Theorem I). If E is connected and $f\colon E \to \mathbf{R}$ is continuous, then for any $p, q \in E$ and any y between $f(p)$ and $f(q)$ there is an $x \in E$ with $f(x) = y$. *You should be able to prove this. Sample problem:* If E is connected and $f\colon E \to \mathbf{R}$ is continuous and bounded. Then for any c with $\inf_{p\in E} f(p) < c < \sup_{p\in E} f(p)$ there is a $b \in E$ with $f(b) = c$. (Note that we are not assuming that f achieves its supremum or infimum.)
   (c) If $f\colon [a,b] \to \mathbf{R}$ is continuous, then for any y between $f(a)$ and $f(b)$ there is an $x \in (a,b)$ with $f(x) = y$. For some sample problems see the last couple of homeworks.

(5) Uniform continuity.
   (a) We have only touched on this, but you should know the definition of uniform continuity. You should also be able to give examples of functions that are not uniformly continuous.
   (b) The big theorem here is that a continuous function on compact space is uniformly continuous. Be able to give an exact statement of this.

(6) Miscellaneous problems to look at;

(a) Page 92 Problem 14.
(b) Page 92, Problem 17.
(c) Let $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ be the unit square in $\mathbf{R}^2$. (a) Show R is connected. (b) Show that no continuous function $f\colon R \to \mathbf{R}$ is one-to-one.
(d) Go over the old homework problems.