Mathematics 172 Homework

We wish to understand how to approximate the solutions to a rate equation of the form

$$P' = f(P)$$

where $f(P)$ is some function. Some examples would be

$$P' = -.3P\left(1 - \frac{30}{P}\right)$$

or, to use a different dependent variable

$$\frac{dy}{dt} = -.5\frac{y}{10 + y^2}.$$

The standard method for dealing with such problems is ***Euler's method***. The basic idea is that if Δt is very small, then

$$P'(t) \approx \frac{P(t + \Delta t)}{\Delta t}.$$

This is nothing more than the definition of the derivative. We can rearrange this to get

$$P(t + \Delta t) \approx P(t) + P'(t)\Delta t.$$

So far we have not used that P is the solution to a rate equation. If We also have $P' = f(P)$ then this last approximate can be rewritten as

$$P(t + \Delta t) \approx P(t) + f(P(t))\Delta t.$$

1. Let

$$P' = 10 - 3P.$$

It $P(2) = 6.2$ estimate $P(2.3)$.

Answer: In this problem $f(P) = 10 - 3P$, $t = 2$, and $\Delta t = .3$. We then have

$$\begin{aligned} P(2.3) &= P(t + \Delta t) \\ &= P(2 + .3) \approx P(2) + f(P(2))(.3) \\ &= 6.2 + (10 - 3 \cdot 6.2)(.3) \\ &= 3.62 \end{aligned}$$

2. If

$$\frac{dN}{dt} = .3N + .9$$

and $N(3) = 2$ estimate $N(3.1)$.

Answer: In this case $f(N) = .3N + .9$ and we have

$$\begin{aligned} N(3.1) &\approx N(3) + f(N(3))(.1) \\ &= 2 + (.3 \cdot 2 + .9)(.1) \\ &= 2.15 \end{aligned}$$

The problem with this is that it only gives good approximations when Δt is small. The way around this, and this is Euler's idea, is to take lots of small steps. So we choose a step size Δt and set

$$t_k = k\Delta t.$$

Thus

$$t_0 = 0, \quad t_1 = \Delta 1, \quad t_2 = 2\Delta t, \quad t_k = 2\Delta t,$$

and so forth. We then have

$$t_{k+1} = t_k + \Delta t.$$

Now given the initial value problem

$$P' = f(P), \qquad P(0) = P_0.$$

Non generate a sequence of values P_1, P_2, P_3, ... by

$$P_1 = P_0 + f(P_0)\Delta t, \quad P_2 = P_1 + f(P_1)\Delta t, \quad P_3 = P_2 + f(P_2)\Delta t \quad \ldots$$

In general to go from one step to the next

$$P_{k+1} = P_k + f(P_k)\Delta t.$$

Then we have the approximation

$$P(t_k) \approx P_k$$

3. Use this method to approximate $P(1)$ by taking five steps of size $\Delta t = .2$ where

$$\frac{dP}{dt} = .2P\left(1 - \frac{P}{30}\right), \qquad P(0) = 20$$

Answer: Here we have $P_0 = 20$ and the method of taking a step is

$$P_{k+1} = P_k + .2P_k\left(1 - \frac{P_k}{30}\right)$$

This gives values in the table

k	t_k	P_k
0	0.0	20.000
1	0.2	20.2666666666667
2	0.4	20.5296829629630
3	0.6	20.7889131047351
4	0.8	21.0442310848229
5	1.0	21.2955207789472

This gives the approximation $P(1) \approx 21.2955$. If we had done steps of size $\Delta t = .1$ the table would be

k	t_k	P_k
0	0.0	20
1	0.1	20.1333333333333
2	0.2	20.2657659259259
3	0.3	20.3972803987348
4	0.4	20.5278599749331
5	0.5	20.6574884843315
6	0.6	20.7861503670312
7	0.7	20.9138306763180
8	0.8	21.0405150808059
9	0.9	21.1661898658449
10	1.0	21.2908419342038

so that this time we get the approximation $P(1) \approx 21.2908$. I had the compute to if for 100 steps of size $\Delta t = .1$ and got $P(1) \approx 21.2866423243231$ and for 1000 steps of size $\Delta t = .001$ $P(1) \approx 21.2862229752552$.

4. If

$$N' = .02N(10 - N) \qquad N(0) = 13$$

(a) Estimate $N(2)$ by taking two steps of size $\Delta t = 1$.

Answer: The table is

k	t_k	N_k
0	0.0	13
1	1.0	12.2200000000000
2	2.0	11.6774320000000

(b) Estimate $N(2)$ by taking four steps of size $\Delta t = .5$.

Answer: The table is

k	t_k	N_k
0	0.0	13
1	0.5	12.6100000000000
2	1.0	12.2808790000000
3	1.5	12.0007670098736
4	2.0	11.7606596226082

(c) Estimate $N(2)$ by taking twenty steps of size $\Delta t = .1$. (Ok I don't really expect anyone to do this by hand. You might try programing your calculator to do it.)

Answer: The table is

k	t_k	N_k
0	0.0	13
1	0.1	12.9220000000000
2	0.2	12.8464838320000
3	0.3	12.7733492149483
4	0.4	12.7024992989132
5	0.5	12.6338423080137
6	0.6	12.5672912112465
7	0.7	12.5027634186949
8	0.8	12.4401805008611
9	0.9	12.3794679290903
10	1.	12.3205548352573
11	1.1	12.2633737890653
12	1.2	12.2078605914660
13	1.3	12.1539540828539
14	1.4	12.1015959648148
15	1.5	12.0507306343198
16	1.6	12.0013050293644
17	1.7	11.9532684851359
18	1.8	11.9065725998832
19	1.9	11.8611711097283
20	2.0	11.8170197717343

(d) Finally for this initial value problem estimate $N(200)$.
Answer: $N(200) \approx 10$.