Mathematics 172 Homework

1. (Verification of Bret's method.) Show that

$$y = -\frac{c}{r} + \left(y_0 + \frac{c}{r}\right) e^{rt}$$

is a solution to the intinal value problem

$$y' = ry + c, \qquad y(0) = y_0$$

where r and c are constants.

2. The special case of this that is relevant to our current topic is that we have a meta population of organism living in a collection of patches with the probability of a populated patch going extent being p_e and the probability of an unpopulated patch being colonized is p_i, then the fraction, f, of the patches that are populated satisfies

$$f' = p_i(1-f) - p_e f = -(p_i + p_e)f + p_i.$$

Therefore Bret's method gives us that the solution is

$$f(t) = \frac{p_i}{p_i + p_e} + \left(f(0) - \frac{p_i}{p_i + p_e}\right) e^{-(p_i+p_e)t}.$$

If we let

$$\widehat{f} = \frac{p_i}{p_i + p_e}$$

be the equilibrium solution, this solution can also be written as

$$f(t) = \widehat{f} + (f(0) - \widehat{f})e^{-(p_i+p_e)t}$$

Use this to solve the following

(a) If $p_e = .5$ and $p_i = .9$, then what is the fraction of patches that are populated in the long run? *Answer:* $\widehat{f} = .9/(.5 + .9) = .642857$ What is the solution with $f(0) = .25$? *Answer:* $f(t) = .642857 + (.25 - .642857)e^{-1.4t} = .642857 - 0.392857e^{-1.4t}$ For this solution what is $f(10)$? *Answer:* $f(10) = 0.642857$

(b) If $p_e = .8$ and $p_i = .4$, then what is the fraction of patches that are populated in the long run? *Answer:* $\widehat{f} = .4/(.4 + .8) = .33333$ What is the solution with $f(0) = .5$? *Answer:* $f(t) = .33333 + (.5 - .33333)e^{-1.2t} = .33333 - .16667e^{-1.4t}$ For this solution what is $f(3)$? *Answer:* $f(10) = 0.337887$

Here is some practice on Euler's method. Recall that this says that for a system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

that if we choose a small Δt and set

$$t_k = k\Delta t, \qquad x_0 = x(0), \qquad y_0 = y(0)$$

and

$$x_{k+1} = x_k + f(x_k, y_k)$$
$$y_{k+1} = y_h + g(x_k, y_k)$$

Then we have the approximations

$$x(t_k) \approx x_k$$
$$y(t_k) \approx y_k$$

3. Using Euler's method and four steps of size $\Delta t = .5$ approximate $x(2)$ and $y(2)$ for the initial value problem

$$\frac{dx}{dt} = .2x + .3y \qquad x(0) = 7$$
$$\frac{dy}{dt} = .4x - .3y \qquad y(0) = 6$$

Answer:

$$x_1 = 8.60000000000000 \qquad y_1 = 6.50000000000000$$
$$x_2 = 10.4350000000000 \qquad y_2 = 7.24500000000000$$
$$x_3 = 12.5652500000000 \qquad y_3 = 8.24525000000000$$
$$x_4 = 5.0585625000000 \qquad y_4 = 9.52151250000000$$

Which gives

$$x(2) \approx 5.0585625000000 \qquad y(2) \approx 9.52151250000000$$