Math 580 Due Wednesday, August 27

1. Some notation

The main subject of this course is the ***integers*** with is just the positive and negative whole numbers along with zero. We will use the notation

$$\mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, 3, \ldots\}$$

for the set of integers and it will be assume that you know the basic algebraic properties of the integers (communicative law, associative law, distributive law, etc.). The ***natural numbers*** is just anther name for the positive integers. These are denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Also important to us will be the ***rational numbers***. These are just the fractions with numerator and denominator both integers and will be denoted by

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

Again I am assuming that you are use to doing algebra with the rational numbers.

2. Review of some algebra

2.1. **Summation notation.** In the most basic case, $a < b$ are integers and $x_a, x_{a+1}, \ldots, x_b$ are numbers, then

$$\sum_{j=a}^{b} x_j = x_a + x_{a+1} + \cdots + x_b.$$

There is nothing special about the index j:

$$\sum_{j=a}^{b} x_j = \sum_{i=a}^{b} x_i = \sum_{k=a}^{b} x_k = \sum_{\alpha=a}^{b} x_\alpha$$

For example we have

$$\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30,$$

$$\begin{aligned}
\sum_{\alpha=-2}^{3} (2\alpha^3 + \alpha) &= (2(-2)^3 + (-2)) + (2(-1)^3 + (-1)) + (2(0)^3 + (0)) \\
&= +(2(1)^3 + (1)) + (2(2)^3 + (2)) + (2(3)^3 + (3)) \\
&= 57,
\end{aligned}$$

and

$$\sum_{k=1}^{100} = 1 + 2 + \cdots + 100 = 5050.$$

To see this last sum let

$$S = 1 + 2 + \cdots + 99 + 100$$

then reversing the order gives

$$S = 100 + 99 + \cdots + 2 + 1.$$

Therefore

$$\begin{aligned} 2S &= S + S \\ &= (1 \;+\; 2 \;+ \cdots + 99 + 100) \\ &+ (100 + 99 + \cdots + 2 + 1) \\ &= (1 + 100) + (2 + 99) + (3 + 98) + \cdots + (99 + 2) + (100 + 1) && \text{(100 terms)} \\ &= 101 + 101 + 101 + \cdots + 101 + 101 && \text{(100 terms)} \\ &= 100 \times 101. \end{aligned}$$

Therefore

$$S = \frac{(100)(101)}{2} = 50(101) = 5050.$$

Problem 1. Use this method to find $\displaystyle\sum_{k=1}^{1{,}000} k = 1 + 2 + \cdots + 1{,}000$. □

Problem 2. More generally use this idea to find a formula for $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n$ where n is a positive integer. □

Here is a basic property of sums.

Proposition 1. *If c_1 and c_2 are constants then*

$$\sum_{k=a}^{b}(c_1 x_k + c_2 y_k) = c_1\left(\sum_{k=a}^{b} x_k\right) + c_2\left(\sum_{k=a}^{b} y_k\right)$$

Problem 3. Prove this. □

We will often want to sum over sets of numbers other than just $k = a, a+1, \ldots, b$. For example

$$\sum_{k \text{ is odd and } 1.5 < k < 3\pi} k^2 = 3^2 + 5^2 + 7^2 + 9^2 = 164$$

and

$$\begin{aligned}\sum_{k \text{ is even and } 31<k\le 50} 1 &= (1+1+\cdots+1) \qquad \begin{pmatrix}\text{number of 1's is number of even}\\ \text{numbers } k \text{ with } 31<k\le 50\end{pmatrix}\\ &= \#\{32,34,36,38,40,42,44,46,48,50\}\\ &= 10.\end{aligned}$$

Problem 4. Compute the following sums:

(a) $\displaystyle\sum_{d \text{ is a positive divisor of } 20} d,$

(b) $\displaystyle\sum_{d \text{ is a positive divisor of } 30} 1,$

(c) $\displaystyle\sum_{d \text{ is a positive divisor of } 32=2^5} 5,$ and

(d) $\displaystyle\sum_{d \text{ is a positive divisor of } 2^n.} 1$

2.2. **Some algebraic identities.** A basic fact about factoring that we all know is

$$x^2 - y^2 = (x+y)(x-y).$$

This and some generalizations will come up several times in this class. A first extension is

$$x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

More generally

Proposition 2. *For any positive integer $n \ge 2$ the identity*

$$x^n - y^n = (x-y)(x^{n-1} + xy^{n-2} + \cdots + x^k y^{n-1-k} + \cdots + xy^{n-2} + y^{n-1}).$$

In summation notation this is

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

Problem 5. Prove this. *Hint:* Here is the argument for $n=5$. Start with the complicated side, distribute and cancel out terms:

$$\begin{aligned}(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) &= x(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &\qquad - y(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &= x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4\\ &\qquad - x^4y - x^3y^2 - x^2y^3 - xy^4 - y^5\\ &= x^5 - y^5.\end{aligned}$$

The general case follows the same pattern. □

Related these is the identity is

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

One way to see this holds is to start with $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ and let $a = x$ and $b = -y$ with gives

$$x^3 + (-y)^3 = (x + (-y))(x^2 + x(-y) + (-y)^2)$$

which clearly simplifies to what we want. Likewise doing the same substitution in $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^n$ gives

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

Proposition 3. *If $n \geq 3$ is an odd positive integer, then*

$$x^n+y^n = (x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\cdots+x^2y^{n-3}-xy^{n-2}+y^{n-1}).$$

Problem 6. Prove this. *Hint:* It is fine with me if you just do this in the case of $n = 7$. □

For the applications we have in mind, we will just generally want to know if some expression factors, and not necessarily care if we have all factors. For example $x^6 + y^6$ factors because we can write

$$x^6+y^6 = (x^2)^3+(y^2)^3 = (x^2+y^2)((x^2)^2-x^2y^2+(y^2)^2) = (x^2+y^2)(x^4-x^2y^2+y^4)$$

Problem 7. Find at least one factorization of the following.

(a) $x^{30} + y^{30}$,
(b) $x^5 + 32y^{10}$, and
(c) $x^{3^n} + 1$ where n is a positive integer. □

3. The Elements of the Calculus of Finite Differences

This section is partly to get some more practice with sums and summation notation. But it is also of interest in its own sake as it allows us to find

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3$$

and related sums.

Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function from the integers, $\mathbb{Z}$, to the real numbers, $\mathbb{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

3.1. **The difference operator and the fundamental theory of summation theory.**

Definition 4. Let $f\colon \mathbb{Z} \to \mathbb{R}$. Then the ***difference***, Δf, of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1)+2) - (3x+2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax + b$	a
cr^x	$c(r-1)r^x$

Problem 8. Verify these. □

Theorem 5 (**Fundamental Theorem of Summation Theory).** *Let $f\colon \mathbb{Z} \to \mathbb{R}$ and let F be an* ***anti-difference*** *of f. That is $\Delta F = f$. Then for $a, b \in \mathbb{Z}$ with $a < b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}
\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b} (F(k+1) - F(k)) \\
&= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k) \\
&= \big(F(a+1) + F(a+2) + \cdots + F(b) + F(b+1)\big) \\
&\qquad - \big(F(a) + F(a+1) + \cdots + F(b-1) + F(b)\big) \\
&= F(b+1) - F(a)
\end{aligned}$$

as required. □

Theorem 5 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a, r and b are constants).

$f(x)$	$F(x)$
$ax+b$	$a\dfrac{x(x-1)}{2}+bx$
ar^x	$\dfrac{ar^x}{1-r}$

Problem 9. Verify these. (You just need to check $F(x+1)-F(x) = f(x)$). □

Problem 10 (Sum of finite geometric series). Use that $\dfrac{ar^x}{1-r}$ as the anti-difference of ar^x and Theorem 5 to show

$$a+ar+ar^2+\cdots+ar^n = \frac{a-ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1-\text{ratio}}.$$

□

3.2. **Falling factorial powers and sums of powers.** For Theorem 5 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 6. For natural number p define the ***falling factorial power*** of $x\in\mathbb{R}$ as $x^{\underline{0}}=1$ and for $p\geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has p terms.) □

For small values of p this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1\\
x^{\underline{1}} &= x\\
x^{\underline{2}} &= x(x-1)\\
x^{\underline{3}} &= x(x-1)(x-2)\\
x^{\underline{4}} &= x(x-1)(x-2)(x-3)\\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

Proposition 7. *If $f(x)=x^{\underline{p}}$ where p is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

Problem 11. Prove this.*Hint:* Here is what the calculation looks like for $p = 4$.

$$\begin{aligned}\Delta x^{\underline{4}} &= (x+1)^{\underline{4}} - x^{\underline{4}} \\ &= (x+1)x(x-1)(x-2) - x(x-1)(x-2)(x-3) \\ &= x(x-1)(x-2)\big((x+1)-(x-3)\big) \\ &= 4x(x-1)(x-2) \\ &= 4x^{\underline{3}}.\end{aligned}$$

The general case is only a bit more complicated. □

Remark 8. The formula should remind you of the formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. □

Proposition 9. *If $f(x) = x^{\underline{p}}$ where p is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of f.*

Problem 12. Prove this as a corollary of Proposition 7 by noting (replace p by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. □

Problem 13. Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) □

Proposition 10. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Remark 11. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

Problem 14. Prove this. Hint: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 5

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 13 to see that $F(1) = 0$. □

Proposition 12. *The equalities*

$$\begin{aligned}x &= x^{\underline{1}} \\ x^2 &= x^{\underline{2}} + x^{\underline{1}} \\ x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\ x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\ x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}\end{aligned}$$

hold.

Problem 15. Verify the first three of these. □

Problem 16. Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

Hint: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 10

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\
&= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}.
\end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^{n} k^3$. □

4. The biomomial theorem

We first recall the definition of the ***factorials***. If n is a non-negative integer $n!$ is defined by

$$0! = 1 \quad \text{and for } n \geq 1 \quad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For small values of n we have

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880

n	$n!$
10	3,628,800
11	39,916,800
12	47,9001,600
13	622,7020,800
14	87,178,291,200
15	1,307,674,368,000
16	20,922,789,888,000
17	3556,87,428,096,000
18	6,402,373,705,728,000
19	121,645,100,408,832,000

n	$n!$
20	2,432,902,008,176,640,000
21	51,090,942,171,709,440,000
22	1,124,000,727,777,607,680,000
23	25,852,016,738,884,976,640,000
24	620,448,401,733,239,439,360,000
25	15,511,210,043,330,985,984,000,000
26	403,291,461,126,605,635,584,000,000
27	10,888,869,450,418,352,160,768,000,000
28	304,888,344,611,713,860,501,504,000,000
29	8,841,761,993,739,701,954,543,616,000,000
30	265,252,859,812,191,058,636,308,480,000,000

This makes it clear $n!$ grows very fast as a function of n.

Problem 17. Show that for $n \geq 10$ that $n! \geq 3.6288(10)^{n-4}$. *Hint:* Use that $10! = 3{,}628{,}800 = 3.6288(10)^6$. For example if $n = 15$

$$\begin{aligned} 15! &= 10!(11)(12)(13)(14)(15) \\ &\geq 10!(10)(10)(10)(10)(10) \\ &= 10!(10)^5 \\ &= 3.6288(10)^6(10)^5 \\ &= 3.6288(10)^{11}. \end{aligned}$$

This idea works in general. $\square$

Remark 13. There is a remarkable formula to approximate $n!$ for large n:

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}.$$

More precisely

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}.$$

This is ***Stirling's formula***. Already for $n = 10$ this gives

$$3.6285(10)^6 < 10! = 3.6288(10)^6 < 3.6288(10)^6$$

which is accurate to four sufficient digits. For $n = 100$ this gives

$$9.3326150(10)^{157} < 100! = 9.33262154(10)^{157} < 9.33262157(10)^{157}$$

which is good to seven sufficient digits. $\square$

Definition 14. Let $n, k \geq 0$ be integers with $0 \leq k \leq n$. Then the ***binomial coefficient*** $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

$\square$

Note that this definition implies

$$\binom{n}{k} = \binom{n}{n-k}.$$

Also we generally do not have to compute $n!$ to find $\binom{n}{k}$ as lots of terms cancel. For example

$$\binom{100}{3} = \frac{100!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98 \cdot 97!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98}{3!} = 161{,}700.$$

Proposition 15. *The following hold*

$$\begin{aligned}
\binom{n}{0} &= \binom{n}{n} = 1, \\
\binom{n}{1} &= \binom{n}{n-1} = n, \\
\binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2}, \\
\binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

Problem 18*.* Prove this. $\square$

More generally we can relate the binomial coefficients to the fall factorial powers of Definition 6.

Proposition 16. *The equality*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

holds.

Problem 19*.* Prove this. $\square$

Here is anther basic property of the binomial coefficients.

Proposition 17. *For $1 \le k \le n$ with k, n integers the equality*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Problem 20*.* Prove this. $\square$

If we put the binomial coefficients in a triangular table (Pascal's triangle):

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ tells us that any entry is the sum of the two entries directly above. This can be used to compute $\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given by:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

One reason the binomial coefficients are important is

Theorem 18 (Binormal Theorem)**.** *For any positive integer n and $x, y \in \mathbb{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We will prove this shortly. So for $n = 5$ we have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Let $x = y = 1$ in this to get

$$
\begin{aligned}
2^5 &= (1+1)^5 \\
&= (1)^5 + 5(1)^4(1) + 10(1)^3(1)^2 + 10(1)^2(1)^3 + 5(1)(1)^4 + (1)^5 \\
&= 1 + 5 + 10 + 10 + 5 + 1,
\end{aligned}
$$

which may not be that interesting of a fact, but the argument lets us see a pattern for something that is interesting

Problem 21. Use this idea to show the sum of the numbers $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is 2^n. That is for all positive integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n. \qquad \square$$

cases

Problem 22. Prove for any positive integer n that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: $(1 - 1) = 0$. $\square$

Here is a bit of practice in using the binomial theorem.

Problem 23. Expand the following:

(a) $(1 + 2x^3)^4$,
(b) $(x^2 - y^5)^3$.