# Mathematics 574 Homework

Here we look the the principle of ***inclusion-exclusion***. The basic formula is that for two finite sets $A$ and $B$

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B).$$

We saw in class this implies

$$\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C).$$

This generalizes to unions of $n$ sets:

**Theorem 1** (Principle of inclusion-exclusion.)**.** *Let $A_1, A_2, \ldots, A_n$ be finite sets. Then*

$$\begin{aligned}
\#(A_1 \cup A_2 \cup \cdots \cup A_n) = &\sum_{1 \le i \le n} \#(A_i) \\
&- \sum_{1 \le i < j \le n} \#(A_i \cap A_j) \\
&+ \sum_{1 \le i < j < k \le n} \#(A_i \cap A_j \cap A_k) \\
&- \sum_{1 \le i < k < k < \ell \le n} \#(A_i \cap A_j \cap A_k \cap A_\ell) \\
&\ \vdots \\
&(-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots i_k \le n} \#(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \\
&\ \vdots \\
&(-1)^{n-1} \#(A_1 \cap A_2 \cap \cdots \cap A_n).
\end{aligned}$$

This looks more complicated than it is. To see the pattern here are what it looks like for small values of $n$

$$\begin{aligned}
\#(A_1 \cap A_2 \cap A_3) = &\#(A_1) + \#(A_2) + \#(A_3) \\
&- \Big(\#(A_1 \cap A_2) + \#(A_1 \cap A_3) + \#(A_2 \cap A_3)\Big) \\
&+ \#(A_1 \cap A_2 \cap A_3).
\end{aligned}$$

$$
\begin{aligned}
\#(A_1 \cap A_2 \cap A_3 \cap A_4) = &\sum_{1 \le i \le 4} \#(A_i) \\
&- \sum_{1 \le i < j \le 4} \#(A_i \cap A_j) \\
&+ \sum_{1 \le i < j < k \le 4} \#(A_i \cap A_j \cap A_k) \\
&- \#(A_1 \cap A_2 \cap A_3 \cap A_4).
\end{aligned}
$$

$$
\begin{aligned}
\#(A_1 \cap A_2 \cap A_3 \cap A_5) = &\sum_{1 \le i \le 5} \#(A_i) \\
&- \sum_{1 \le i < j \le 5} \#(A_i \cap A_j) \\
&+ \sum_{1 \le i < j < k \le 5} \#(A_i \cap A_j \cap A_k) \\
&- \sum_{1 \le i < j < k < \ell \le 5} \#(A_i \cap A_j \cap A_k \cap A_\ell) \\
&+ \#(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5).
\end{aligned}
$$

To use notation a little more like that of the book, given the sets $A_1, A_2, \ldots, A_n$ for $1 \le k \le n$ define

$$
S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \#(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}).
$$

Then the principle of inclusion-exclusion becomes

$$
\#(A_1 \cup A_2 \cup \cdots \cup A_n) = S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n-1} S_n
$$

**1.** How many terms are there in the sum defining $S_k$? *Hint:* For $k = 3$ we are looking at the sum

$$
S_3 = \sum_{1 \le i < j < k \le n} \#(A_j \cap A_j \cap A_k).
$$

So the number of terms is the same as the number of elements in the set of 3-tuples $\{(i, j, k) : 1 \le i < j < k \le n\}$. But this has the same size as the set of all size three subsets of $\{1, 2, \ldots, n\}$. That is the number of terms in the sum is the same as the number of size three subsets of a size $n$ set and we know what this number is. $\square$

**2.** If we have three sets $A, B, C$ each of size 10 and $\#(A \cap B) = \#(A \cap C) = \#(B \cap C) = 1$ then what can be said about $\#(A \cup B \cup C)$? $\square$

**3.** If $X$ is a set of size $r$ and $Y$ is a set of size $n$, then what is the size of the set, $\mathcal{F}$ of all functions $f : X \to Y$? *Hint:* This is a review question. The answer is $\#(\mathcal{F}) = n^r$.

**4.** Let $X$ be a set of size $r$ and $Y$ a set of size $n$. Let $Y = \{y_1, y_2, \dots, y_n\}$. For $y_j \in Y$, let $\mathcal{F}(y_j)$ be the set of functions $f\colon X \to Y$ such that $y_j$ is *not* in the range of $f$. Show that

$$\#(\mathcal{F}(y_j)) = (n-1)^r.$$

*Hint:* Let $Y^* = Y \setminus \{y_j\}$ and explain why $\mathcal{F}(y_j)$ is just the same set as all functions $f\colon X \to Y^*$ and use the last problem. $\square$.

**5.** With the notation of the last problem show that if $y_i, y_j \in Y$ and $y_i \neq y_j$, then

$$\#(\mathcal{F}(y_i) \cap \mathcal{F}(y_j)) = (n-2)^r.$$

$\square$

**6.** Let $X$ be a set of size $r$ and $Y = \{y_1, y_2, y_3\}$ a set of size 3. Let $\mathcal{M}$ be the set of functions $f\colon X \to Y$ that are *not* onto.

(a) Explain why $\mathcal{M} = \mathcal{F}(y_1) \cup \mathcal{F}(y_2) \cup \mathcal{F}(y_3)$.
(b) Explain why $\mathcal{F}(y_1) \cap \mathcal{F}(y_2) \cap \mathcal{F}(y_3) = \varnothing$
(c) Use the principle of inclusion-exclusion to find $\#(\mathcal{M})$.
(d) As $\mathcal{M}$ is the set of functions that are not surjections, the size of the set of surjections is

$$\#(\mathcal{F}) - \#(\mathcal{M}).$$

Use your answer to Part (c) to show that when $\#(Y) = 3$

$$\text{Number of surjections } f\colon X \to Y = 3^r - 3 \cdot 2^r + 3$$

We now able to prove the following.

**Theorem 2.** *Let $X$ and $Y$ be finite sets with $\#(X) = r$ and $\#(Y) = n$ with $r \geq n$. Let $\mathcal{S}$ be the set of surjections $f\colon X \to Y$. Then*

$$\#(\mathcal{S}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r.$$

**7.** Prove this along the following line. First let $\mathcal{M}$ be the set of functions $f\colon X \to Y$ that are **not** surjections and $\mathcal{F}$ the set of all functions form $X$ to $Y$. we have

$$\#(\mathcal{S}) = \#(\mathcal{F}) - \#(\mathcal{M}) = n^r - \#(\mathcal{M})$$

So, as in the last problem, it is enough to find the size of $\mathcal{M}$. As in that problem let $Y = \{y_1, y_2, \dots, y_n\}$ and set

$$\begin{aligned} \mathcal{F}(y_i) &= \text{set of functions in } \mathcal{F} \text{ that do not have } y_i \text{ in their range.} \\ &= \{f : f \text{ is function } f\colon X \to Y \setminus \{y_i\}\}. \end{aligned}$$

(a) Show

$$\mathcal{M} = \mathcal{F}(y_1) \cup \mathcal{F}(y_2) \cup \cdots \cup \mathcal{F}(y_n).$$

(b) Show

$$\#(\mathcal{F}(j_i)) = (n-1)^r.$$

(c) More generally show that if $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$ are distinct elements of $Y$ then

$$\mathcal{F}(y_{i_1}) \cap \mathcal{F}(y_{i_2}) \cap \cdots \cap \mathcal{F}(y_{i_k}) = \{f : f \text{ is function } f \colon X \to Y \setminus \{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\}\}$$

and therefore

$$\#(\mathcal{F}(y_{i_1}) \cap \mathcal{F}(y_{i_2}) \cap \cdots \cap \mathcal{F}(y_{i_k})) = (n-k)^r.$$

(d) Let

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \#(\mathcal{F}(y_{i_1}) \cap \mathcal{F}(y_{i_2}) \cap \cdots \cap \mathcal{F}(y_{i_k})).$$

Show

$$S_k = \binom{n}{k}(n-k)^r.$$

(e) Now use inclusion-exclusion to show

$$\#(\mathcal{M}) = S_1 - S_2 + S_3 - \cdots = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)^r.$$

(f) Finally use $\#(\mathcal{S}) = n^r - \#(\mathcal{M})$ to complete the proof of Theorem 2. □

**8.** Consider the case of $X = Y$ in Theorem 2. Then $\#(X) = \#(Y) = n$. Also, in this case, a function is a surjection if and only if it is an injection. Therefore the number of surjections is the same as the number of injections. Use to prove the identity

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.$$

□

**9.** How many permutations of $a, b, c, d, \ldots, x, y, z$ do not contain any of the words "quick", "red", or "fox"? *Hint:* Use inclusion-exclusion to first find the number that contain at least one of these words. □

**10.** How many permutations of $a, b, c, d, \ldots, x, y, z$ do not contain any of the words "very", "bad", or "boy"? *Hint:* This slightly trickier than the last problem because two of the words ("bad" and "boy") have a letter in common. □

**11.** An office temp has a one day job in a office where he does not care if he makes a good impression. He is suppose to put 5 letters in 5 envelopes so that the address on the letter matches the address on the envelope. He does this at random. We now figure out the number of ways that he can do this so that at least one letter goes into the correct envelope. We can think of a pairing of letters and envelopes as a bijection, $f$, of $\{1, 2, 3, 4, 5\}$. That is $f(j) = k$ means that letter $j$ went into envelope $k$ (i.e. $f(3) = 1$ means that letter 3 when into envelope 1, $f(4) = 4$ means that the letter 4 went into the correct envelope). Let $\mathcal{P}$ be the set of all permutations of $\{1, 2, 3, 4, 5\}$. Let

$$\mathcal{P}(i) = \{f \in \mathcal{P} : f(i) = i\}.$$

(a) Let $\mathcal{G}$ be the set of functions such that $f(i) = i$ for at least one value of $i$. This corresponds there being at least one letter that does into the correct envelope. Explain why

$$\mathcal{G} = \mathcal{P}(1) \cup \mathcal{P}(2) \cup \mathcal{P}(3) \cup \mathcal{P}(4) \cup \mathcal{P}(5).$$

(b) Show

$$\#\mathcal{P}(i) = 4!$$

(c) Show that if $i \neq j$, then

$$\#(\mathcal{P}(i) \cap \mathcal{P}(j)) = 3!$$

(d) If $i, j, k$ are distinct compute $\#(\mathcal{P}(i) \cap \mathcal{P}(j) \cap \mathcal{P}(k))$.

(e) If $i, j, k, \ell$ are distinct compute $\#(\mathcal{P}(i) \cap \mathcal{P}(j) \cap \mathcal{P}(k) \cap \mathcal{P}(\ell))$.

(f) Compute $\#(\mathcal{P}(1) \cap \mathcal{P}(2) \cap \mathcal{P}(3) \cap \mathcal{P}(4) \cap \mathcal{P}(5))$.

(g) Now use inclusion-exclusion to compute $\#(\mathcal{G})$. $\square$