Mathematics 300

Quiz 18 **Name:** Answer Key

You must show your work to get full credit.

1. We know from our study of sets that

(1) $$\overline{B \cap C} = \overline{B} \cup \overline{C}.$$

Use this and induction to show that if $A_1, A_2, \ldots, A_n$ are sets in some universal set A that

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n.$$

Solution. We use as the base case $n = 2$ where the statement becomes

$$\overline{A_1 \cap A_2} = \overline{A}_1 \cup \overline{A}_2$$

which is $\overline{B \cap C} = \overline{B} \cup \overline{C}$ with $B = A_1$ and $C = A_2$.

For the induction step assume that

(2) $$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A}_1 \cup \overline{A}_2 \cup \cdots \cup \overline{A}_n.$$

and use this to show that the result holds for with n repalaced by $n + 1$.

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}} &= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}} \\
&= \overline{(A_1 \cap A_2 \cap \cdots \cap A_n)} \cap \overline{A}_{n+1} \qquad \text{(Using (1) with } B = (A_1 \cdots \cap A_n) \text{ and } C = A_{n+1}) \\
&= \overline{A}_2 \cup \cdots \cup \overline{A}_n \cup \overline{A}_{n+1} \qquad \text{(Using Equation (2)).}
\end{aligned}$$

This completes the induction step. □

2. For any integer $n \geq 1$, we have $3 \mid (n^3 + 5n + 6)$.

Solution. The base case is $n = 1$. Then $n^3 + 5n + 6 = 1 + 5 + 6 = 12$ and $3 \mod 12$, so the base case holds.

For the induction step assume that $3 \mid (n^3 + 5n + 6)$. Then for some integer k we have

$$n^3 + 5n + 6 = 3k.$$

Now

$$\begin{aligned}
(n + 1)^3 + 5(n + 1) + 6 &= n^3 + 3n^2 + 3n + 1 + 5n + 5 + 6 \\
&= (n^3 + 5n + 6) + 3n^2 + 3n^2 + 3n + 6 \\
&= 3k + 3(n^2 + n + 2) \\
&= 3(k + n^2 + n + 2) \\
&= 3\ell
\end{aligned}$$

where ℓ is the integer $\ell = k + n^2 + n + 2$. Therefore $2 \mid (n + 1)^3 + 5(n + 1) + 6)$ which completes the induction. □