Review for Test 2.

In the following Homework 3 and Homework 4 refer to the homework sets class webpage. *Rosenlicht* refers to our text.

(1) *Metric spaces.* This is one of the main topics we have covered since the last test. You should know the definition of a ***metric space***, ***open ball***, ***closed ball***, ***open set***, and ***closed set***. Showing that some set is open or closed is a reasonable question. Here are some questions that would be reasonable:
   (a) Let E be a metric space and $a, b \in E$ with $a \neq b$. Show that the set $U = \{x \in E : d(a, x) > d(b, x)\}$ is open.
   (b) In $\mathbb{R}^2$ show the set $U = \{(x, y) : x > 0 \text{ and } y > 0\}$ is open.
   (c) Let E be a metric space and $f\colon E \to \mathbb{R}$ a functions such that $|f(p) - f(q)| \leq 3d(p, q)$ for all $p, q \in E$. Then $U = \{x \in E : f(x) > 42\}$ is open.

(2) *Adherent points and closed sets.* You should know the definition of ***adherent point*** and that a set is closed if and only if it contains all its adherent points (cf. Homework 4 Theorem 38).

(3) *Limits of sequences.* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space. You need to know the definition of $\lim_{n\to\infty} p_n = p$. We did a good deal with limits.
   (a) The characterization of closed sets as those that contain the limits of their convergent sequences (Homework 4, Theorem 39).
   (b) You should be able to prove things such as convergent series in a metric space are bounded. As an example be able so show that if $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} x_n = 13$, then are are only finitely many n such that $x_n > 17$.
   (c) For polynomials and rational functions $f\colon \mathbb{R} \to \mathbb{R}$ we proved that if $\lim_{n\to\infty} p_n = p$ that $\lim_{n\to\infty} f(p_n) = f(p)$. You should understand how these proofs work. Here are a some sample problems:
      (i) Let E be a metric space and $a \in E$. Let $\lim_{n\to\infty} p_n = p$. Show that $\lim_{n\to\infty} d(p_n, a) = d(p, a)$, $\lim_{n\to\infty} d(p_n, a)^2 = d(p, a)^2$, and if $p \neq a$, that $\lim_{n\to\infty} 1/d(p_n, a) = 1/d(p, a)$.
      (ii) If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of real numbers with $\lim_{n\to\infty} x_n = a > 0$, then $\lim_{n\to\infty} \sqrt[3]{x_n} = \sqrt[3]{a}$.

(4) *Cauchy sequences and completeness.* You should definitely know the definitions of a ***Cauchy sequence*** and a ***complete metric space***. One of the big results is that the real numbers are a complete metric space. You should look at the proof of in Homework 4. I consider asking for the proofs of Proposition 45, Proposition 47, Theorem 48, Theorem 49, Proposition 50, and Theorem 51 as fair game. Looking at the proof that $\mathbb{R}^n$ is complete would also be a good idea.

(5) *The Bolzano-Weierstrass Theorem and sequential compactness.* You should be able to prove Bolzano-Weierstrass Theorem (Homework 4, Theorem 58) and know the definition of ***sequential compactness***.

(6) *Compactness and the Lebesgue Covering Theorem.* You should know the definitions of ***open cover***, ***subcover***, and ***compact space***. You should also know the statement of the ***Lebesgue Covering Theorem*** Homework 4, Theorem 67.

(7) Here are some practice problems.

  (a) Let $a \geq 1$ be a real number define a sequence $x_0, x_1, x_2, \ldots$ by $x_0 = \sqrt{a}$, and $x_{n+1} = \sqrt{a + x_n}$ for $n \geq 0$. That is

$$x_0 = \sqrt{a}$$
$$x_1 = \sqrt{a + \sqrt{a}}$$
$$x_2 = \sqrt{a + \sqrt{a + \sqrt{a}}}$$
$$x_3 = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}}$$

  Show that $A = \lim n \to \infty x_n$ exists and find the limit. Prove your result.

  (b) In *Rosenlicht* look at problems 15 and 24 on pages 62–64.

(8) Surprise mystery questions.