Review for Test 3.

In the following NOTES refers to the file Notes

`http://ralphhoward.github.io/Classes/Fall2018/554/notes.pdf`

on the class webpage:

`http://ralphhoward.github.io/Classes/Fall2018/554/`

Rosenlicht refers to our text.

(1) The main topic we have covered is the continuity of continuous functions. If $f\colon E \to E'$ is a map between metric spaces you should know the definition of f being continuous at the point $p_0 \in E$. Some things related to this definition you should know or be able to do are
   (a) First off knowing the definition is important.
   (b) You will have to do at least one ε–δ proof for a concrete function. Examples would be showing that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is continuous at $x = a$, and the function $g(x) = 1/x$ (these are worked below), and Problem 4.9 in NOTES are examples.
   (c) For functions $f, g\colon E \to \mathbb{R}$ from a metric space E to the real numbers, we have a theorem (NOTES Theorem 4.5) about when linear combinations, products, and quotients of these functions are continuous at a point. You should be able state and prove these results. Along these lines you should be able to prove that polynomials are continuous (NOTES Proposition 4.7). Also being able to prove that Lipschitz functions are continuous might come up.
   (d) Understand that the composition of continuous functions is continuous (NOTES Theorem 4.8) and how to prove this result.

(2) Once we covered the basics about continuity we then showed that continuity can be characterized in terms of limits of sequence and also in terms of preimages of open and closed sets. That is Theorems 4.11 and 4.13. These are also results where you should know both the statements and proofs.
   (a) Some problems to look at would be Problem 4.21 in NOTES and Problems 2 and 4 page 91 of *Rosenlicht*.

(3) We then studied continuous images of connected sets and the intermediate value theorem.
   (a) You should certainly know the statement and proofs of Theorems 4.16 and 4.17 in NOTES . (And I hope that the ease of these proofs convinces you that relating continuity to preimages of open sets makes this easy to understand.)
   (b) You should know the standard form of the Intermediate Value Theorem given by Theorem 4.19 in the notes. There will very likely be a problem like one of the parts of Problem 4.24 of the notes where you have to use the Intermediate Value Theorem to show that some equation or anther has a solution.

(c) There could also be a problem like Problem 4.31 in NOTES where you show that some set is connected by showing that it is the union of overlapping continuous images of continuous sets.

(d) Here is a practice problem: Let E be a metric space and $S \subseteq E$ with $S \neq \varnothing$ and $S \neq E$. Assume that the function

$$f(x) = \begin{cases} 1, & x \in S; \\ 0, & x \notin S. \end{cases}$$

is continuous. Show that E is not connected.

Problem 1. Show that $f(x) = x^3$ is continuous at $x = a$.

Solution. First note that

$$\begin{aligned} |x^3 - a^3| &= |(x-a)(x^2 + ax + a^2)| \\ &= |x-a||x^2 + ax + a^2| \\ &\leq |x-a|\big(|x|^2 + |a||x| + |a|^2\big) \end{aligned}$$

Assume that $|x - a| < 1$, then, doing a calculation we have done several times,

$$|x| = |a + (x-a)| \leq |a| + |x-a| < |a| + 1.$$

Therefore if $|x-a| < 1$ we have

$$\begin{aligned} |x^3 - a^3| &\leq |x-a|\big(|x|^2 + |a||x| + |a|^2\big) \\ &\leq |x-a|\big((|a|+1)^2 + |a|(|a|+1) + |a|^2\big) \\ &\leq |x-a|\big(3|a|^2 + 3|a| + 1\big) \end{aligned}$$

Let $\varepsilon > 0$, then set

$$\delta = \min\left\{1, \frac{\varepsilon}{3|a|^2 + 3|a| + 1}\right\}$$

Then $|x-a| \leq \delta$ implies

$$\begin{aligned} |x^3 - a^3| &\leq |x-a|\big(3|a|^2 + 3|a| + 1\big) && \text{(as } |x-a| < 1) \\ &\leq \delta\left(3|a|^2 + 3|a| + 1\right) && \text{(as } |x-a| < \delta) \\ &< \frac{\varepsilon}{3|a|^2 + 3|a| + 1}\big(3|a|^2 + 3|a| + 1\big) && \text{(definition of } \delta) \\ &= \varepsilon. \end{aligned}$$

Thus $|x-a| < \delta$ implies $|x^3 - a^3| < \varepsilon$ and therefore $f(x) = x^3$ is continuous at $x = a$. $\square$

Problem 2. Show the function $f(x) = 1/x$ is continuous at $x = b$ where $b > 0$.

Solution. We first do a preliminary calculation

$$\begin{aligned} |f(x) - f(b)| &= \left| \frac{1}{x} - \frac{1}{b} \right| \\ &= \frac{|b-x|}{b|x|} \end{aligned}$$

Assume that

$$|x - b| < \frac{b}{2}.$$

Then

$$x = b + (x - b) \geq b - |x - b| \geq b - \frac{b}{2} = \frac{b}{2}.$$

This implies $x > 0$ and

$$\frac{1}{x} < \frac{1}{\left(\frac{b}{2}\right)} = \frac{2}{b}.$$

Using this in our preliminary calculation for $|f(x) - f(a)|$ we see that $|x - b| < b/2$ implies

$$|f(x) = f(b)| \leq \frac{|b-x|}{b|x|} \leq \frac{2|b-x|}{b^2}.$$

Now let $\varepsilon > 0$ and set

$$\delta = \min\left\{ \frac{b}{2}, \frac{b^2}{2}\,\varepsilon \right\}.$$

Then $|x - b| < \delta$ implies

$$\begin{aligned} |f(x) - f(b)| &\leq \frac{2|b-x|}{b^2} && \text{(form above)} \\ &\leq \frac{2}{b^2}\,\delta && \text{(as } |x-b| < \delta\text{)} \\ &< \frac{2}{b^2}\,\frac{b^2}{2}\,\varepsilon && \text{(definition of } \delta\text{)} \\ &= \varepsilon. \end{aligned}$$

This shows $|x - b| < \delta$ implies $|f(x) - f(b)| < \varepsilon$ and therefore f is continuous at $x = b$. $\square$