Mathematics 242 Homework.

Here is some review on series. A ***power series*** centered at x_0 is a series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

This has a ***radius of convergence*** R so that

- When $|x - x_0| < R$ the series for $f(x)$ converges.
- When $|x - x_0| > R$ the series for $f(x)$ diverges.

In most of the examples we will see in this class the radius of convergence can be found by use of the ratio test.

Theorem 1 (Ratio Test)**.** *Let*

$$S = \sum_{k=0}^{\infty} c_k$$

be a series of numbers and assume that

$$\mathsf{ratio} = \lim_{k \to \infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exits. Then

- *if* $\mathsf{ratio} < 1$ *the series converges absolutely.*
- *If* $\mathsf{ratio} > 1$ *the series diverges.*

Example 2. Find the radius of convergence of the series

$$f(x) = \sum_{k=0}^{\infty} \frac{2^k (x-5)^k}{k(k+1)}$$

We compute the ratio

$$\begin{aligned} \mathsf{ratio} &= \lim_{k \to \infty} \left| \left(\frac{2^{k+1}(x-5)^{k+1}}{(k+1)(k+1+1)} \right) \left(\frac{k(k+1)}{2^k (x-5)^k} \right) \right| \\ &= \lim_{k \to \infty} \frac{2|x-5|k}{k+2} \\ &= 2|x-5| \end{aligned}$$

Therefore, by the ratio test, the series converges when

$$\mathsf{ratio} = 2|x - 5| < 1$$

that is when

$$|x - 5| < \frac{1}{2}.$$

Therefore the radius of convergence is $R = \frac{1}{2}$ and the series converges absolutely on the interval

$$(5 - R, 5 + R) = (5 - 1/2, 5 + 1/2) = (4.5, 5.5). \qquad \square$$

Example 3. Find the radius of convergence of

$$h(x) = \sum_{k=0}^{\infty} k! x^k$$

Again we compute the ratio

$$\begin{aligned}
\textsf{ratio} &= \lim_{k\to\infty} \left| \frac{(k+1)! x^{k+1}}{k! x^k} \right| \\
&= \lim_{k\to\infty} (k+1)|x| \\
&= \begin{cases} \infty, & x \neq 0; \\ 0, & x = 0. \end{cases}
\end{aligned}$$

So in this case the radius of convergence is $R = 0$ and the only value where the series converges is $x = 0$. $\square$

Example 4. Find the radius of convergence of

$$g(x) = \sum_{k=0}^{\infty} \frac{3^k (x+2)^{2k}}{k!}.$$

Yet again we compute the ratio

$$\begin{aligned}
\textsf{ratio} &= \lim_{k\to\infty} \left| \left(\frac{3^{k+1}(x+2)^{2(k+1)}}{(k+1)!} \right) \left(\frac{k!}{3^k (x+2)^{2k}} \right) \right| \\
&= \lim_{k\to\infty} \frac{3|x+2|^2}{k+1} \\
&= 0.
\end{aligned}$$

Therefore $\textsf{ratio} = 0$ for all x and thus the series converges for all x. In this case we say that $R = \infty$, that is the radius of convergence is infinite. $\square$

Problem 1. Find the radius of convergence for each of the following series.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{k(x-1)^{2k}}{4^k}$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2 x^{3k}}{k!}$.

(c) $\displaystyle\sum_{k=0}^{\infty} k^{100} x^k$.

Solution. (a) We have

$$\textsf{ratio} = \lim_{k\to\infty} \left| \frac{(k+1)(x-1)^{2(k+1)}}{4^{k+1}} \frac{4^k}{k(x-1)^{2k}} \right| = \frac{|x-1|^2}{4} < 1$$

when $|x-1| < 2$. Therefore $R = 2$.

(b) This time

$$\text{ratio} = \left| \frac{(k+1)^2 x^{3(k+1)}}{(k+1)!} \frac{k!}{k^2 x^{3k}} \right| = 0 < 1$$

and therefore this converges for all x, that is $R + \infty$.

(c) Again

$$\text{ratio} = \lim_{k \to \infty} \left| \frac{k^{101} x^{k+1}}{k^{100} x^k} \right| = |x|$$

and so $R = 1$. □

We now want to use series to get solutions to differential equations. To start we need some facts about taking derivatives of series. Let

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

Then, formally, the derivative of this can be obtained by taking the derivatives term wise

$$f'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1}$$

Note we can drop the $k = 0$ term as $k a_k (x - x_0)^{k-1} = 0$ when $k = 0$. Thus

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$

We would now like to have exponents of $(x - x_0)$ be k. If we replave k by $k + 1$ we get

$$f'(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k.$$

We will also want to multiply series by powers of x. To start note

$$x f(x) = x \sum_{k=0}^{\infty} a_k (x - x_0)^k. = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+1}.$$

This time we replace k by $k - 1$ and note that the smallest power of $(x - x_0)$ in the sum is $(x - x_0)$ so that

$$x f(x) = \sum_{k=1}^{\infty} a_{k-1} (x - x_0)^k.$$

More generally we get

$$x^m f(x) = x^m \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{k=0}^{\infty} a_k (x - x_0)^{k+m} = \sum_{k=m}^{\infty} a_{k-m} (x - x_0)^k.$$

Let us summarize our formulas to date

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} a_k (x - x_0)^k \\
f'(x) &= \sum_{k=0} (k+1) a_{k+1} (x - x_0)^k \\
x f(x) &= \sum_{k=1}^{\infty} a_{k-1} (x - x_0)^k \\
x^2 f(x) &= \sum_{k=2}^{\infty} a_{k-2} (x - x_0)^k \\
x^3 f(x) &= \sum_{k=3}^{\infty} a_{k-3} (x - x_0)^k \\
x^m f(x) &= \sum_{k=m}^{\infty} a_{k-m} (x - x_0)^k
\end{aligned}
$$

Also we know from our calculus class that

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

which tells use that

$$a_0 = f(x_0).$$

Now let us solve a differential equation: find the series solution to

$$y' + (1+x)y = 0, \qquad y(0) = 3.$$

Since the initial condition is given at $x = 0$ we will be expanding about 0. So assume

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Then

$$
\begin{aligned}
y' &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k \\
xy &= \sum_{k=1}^{\infty} a_{k-1} x^k
\end{aligned}
$$

Therefore

$$y' + (1+x)y = y' + y + xy = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k + \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Now what we have to pay attention to is that the sum do not all start at the same index. The first two start at $k = 0$, while the last one starts at $x = 1$. So when combining them the $k = 0$ term has to be treated separately:

$$0 = y' + (1 + x)y = (0 + 1)a_{0+1} + a_0 + \sum_{k=1}^{\infty} \big((k + 1)a_{k+1} + a_k + a_{k-1}\big)x^k.$$

This leads to (for the $k = 0$ term)

$$a_1 + a_0 = 0$$

and for the $k \geq 1$ terms

$$(k + 1)a_{k+1} + a_k + a_{k-1} = 0$$

Let us rewrite these as

$$\begin{aligned} a_1 &= -a_0 \\ a_{k+1} &= -\left(\frac{a_k + a_{k-1}}{k + 1}\right). \end{aligned}$$

Now since $y(0) = 3$ we have

$$a_0 = y(0) = 3.$$

Example 5. For the initial value problem

$$y' + (1 + x)y = 0, \qquad y(0) = 3$$

find

(a) The general recursion on the coefficients,
(b) The first six coefficients $a_0, a_1, a_2, a_3, a_4, a_5$,
(c) The first six terms of the series for y

Solution: Let $y = \sum_{k=0}^{\infty} a_k x^k$ be as above. Then have done done almost all the work above. By the general recursion we the formula for a_{k+1} in terms of previous terms:

$$\begin{aligned} a_1 &= -a_0 \\ a_{k+1} &= -\left(\frac{a_k + a_{k-1}}{k + 1}\right). \end{aligned}$$

We can now find the first several coefficients. We know $a_0 = 3$ thus

$$\begin{aligned}
a_1 &= -a_0 = -3 \\
a_2 &= -\left(\frac{a_1 + a_0}{2}\right) = -\left(\frac{-3+3}{2}\right) = 0 \\
a_3 &= -\left(\frac{a_2 + a_1}{3}\right) = -\left(\frac{0+(-3)}{3}\right) = 1 \\
a_4 &= -\left(\frac{a_3 + a_2}{4}\right) = -\left(\frac{1+0}{4}\right) = -\frac{1}{4} \\
a_5 &= -\left(\frac{a_4 + a_3}{5}\right) = -\left(\frac{(-1/4)+1}{5}\right) = -\frac{3}{20}
\end{aligned}$$

Thus the first several terms of the series are

$$\begin{aligned}
y &= a_1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 a_5 x^5 + \cdots \\
&= 3 - 3x + x^3 - \frac{1}{4}x^4 - \frac{3x^5}{20} + \cdots
\end{aligned}$$

Problem 2. For the following two initial value problems

(i) The general recursion on the coefficients,
(ii) The first six coefficients $a_0, a_1, a_2, a_3, a_4, a_5$,
(iii) The first six terms of the series for y,

(a) $y' + (3 - 2x)y = 0, \quad y(0) = 12.$
(b) $y' + 2xy = 1 + x, \quad y(0) = 9.$

Solution. (a) Let

$$y = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y' + (3-2x)y = a_1 + 3a_0 + \sum_{k=1}^{\infty} \Big((k+1)a_{k+1} + 3a_k - 2a_{k-1}\Big)x^k = 0$$

therefore

$$\begin{aligned}
a_0 + 3a_0 &= 0 \\
(k+1)a_{k+1} + 3a_k - 2a_{k-1} &= 0 \qquad\qquad (\text{for } k \geq 10)
\end{aligned}$$

Therefore

$$\begin{aligned}
a_0 &= 12 \\
a_1 &= -3a_0 = -36 \\
a_{k+1} &= \frac{-3a_k + 2a_{k-1}}{k+1}
\end{aligned}$$

This can be used to compute

$$\begin{aligned} a_0 &= 12 \\ a_1 &= -36 \\ a_2 &= 66 \\ a_3 &= -90 \\ a_4 &= 201/2 \\ a_5 &= -963/10 \\ a_6 &= 1633/20 \end{aligned}$$

and thus

$$y = 12 - 26x + 66x^2 - 90x^3 + \frac{201}{2}x^4 - \frac{963}{10}x^5 + \cdots$$

(b) Again let

$$y = \sum_{k=0}^{\infty} a_k x^k$$

Then

$$y' + 2xy = a_1 + \sum_{k=1}^{\infty} \big((k+1)a_{k+1} + 2a_{k-1}\big)x^k = 1 + x$$

which leads to

$$\begin{aligned} a_1 &= 1 \\ 2a_2 + 2a_0 &= 1 \\ (k+1)a_{k+1} + 2a_{k-1} &= 0 \qquad\qquad (\text{for } k \geq 2) \end{aligned}$$

Thus

$$\begin{aligned} a_0 &= 9 \\ a_1 &= 1 \\ a_2 &= \frac{1 - 2a_0}{2} = \frac{-17}{2} \\ a_{k+1} &= \frac{-2a_{k-1}}{k+1} \qquad\qquad (\text{for } k \geq 2), \end{aligned}$$

and

$$\begin{aligned} a_0 &= 9 \\ a_1 &= 1 \\ a_2 &= -17/2 \\ a_3 &= -2/3 \\ a_4 &= 17/4 \\ a_5 &= 4/15 \\ a_6 &= -17/12 \end{aligned}$$

giving the series as

$$y = 9 + x + -\frac{17}{2}x^2 - \frac{2}{3}x^3 + \frac{17}{4}x^4 + \frac{4}{15}x^5 + \cdots$$

$\square$

Problem 3. For second order equation

$$y'' - xy' - 2y = 0$$

(i) The general recursion on the coefficients,
(ii) The first five coefficients a_0, a_1, a_2, a_3, a_4
(iii) The first five terms of the series for y,

(a) When the initial conditions are $y(0) = 1$ and $y'(0) = 0$,
(b) When the initial conditions are $y(0) = 1$ and $y'(0) = 1$,

Solution. As usual start with

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Then for either parts (a) or (b) we have

$$y'' - xy' - 2y = 2a_2 - 2a_0 + \sum_{k=1}^{\infty} \big((k+2)(k+1)a_{k+2} - (k+2)a_k\big)x^k = 0$$

Whence

$$\begin{aligned} 2a_2 - 2a_0 &= 0 \\ (k+2)(k+1)a_{k+2} - (k+2)a_k &= 0 \qquad\qquad \text{(for } k \geq 1) \end{aligned}$$

which gives

$$\begin{aligned} a_2 &= a_0 \\ a_{k+2} &= \frac{a_k}{k+1} \qquad\qquad \text{(for } k \geq 1). \end{aligned}$$

(a) In this case we have $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$. Using the recursion we have

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 0 \\ a_4 &= 1/3 \\ a_5 &= 0 \\ a_6 &= 1/15 \end{aligned}$$

and

$$y = 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{15}x^6 + \cdots$$

(b) This time $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$ giving

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \\ a_2 &= 1 \\ a_3 &= 1/2 \\ a_4 &= 1/3 \\ a_5 &= 1/8 \\ a_6 &= 1/15 \end{aligned}$$

and

$$y = 1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{1}{8}x^5 + \cdots$$

$\square$