Mathematics 242 Homework.

In this homework we will mostly go over topics we have covered.

First we look at ***autonomous equations***. These are equations of the for

$$\frac{dy}{dx} = f(y).$$

That is equations where the independent variable does not appear explicitly. These equations are separable:

$$\frac{dy}{dx} = f(y)$$

can be separated into

$$\frac{dy}{f(y)} = dx$$

which can be integrated as

$$\int \frac{dy}{f(y)} = \int dx = x + c.$$

This integral and the resulting formula may be quite complicated. Fortunately these equations can be analyzed by finding the ***critical points***. These are the constants k with $f(k) = 0$. Then the constant function $y = k$ is a solution to the equation. That is the critical are the constant solutions to the equation. These, along with using the sign of $y' = f(y)$ to determine if the solution is increasing or decreasing, can be used to make sketches of the solutions. Some examples of this are at

`http://ralphhoward.github.io/Classes/Fall2020/242/Lesson_1/`

Problem 1. For the following equations find the critical points, sketch some solutions to the equation on the same axis as the critical solutions and determine if the critical points are stable or unstable.

(a) $y' = 30 - .5y$

(b) $\dfrac{du}{dt} = .05u\left(1 - \dfrac{u}{100}\right).$

(c) $y' = .1y(y-2)(5-y)$ □

Problem 2. Let y be the solution to $y' = 30 - .5y$ with $y(0) = 50$.

(a) Use your graphs from Problem 1 (a) to estimate $y(100)$.

(b) This equation is both separable and first order linear so we can solve it explicitly. Using the method of your choice, find the solution with $y(0) = 50$.

(c) What is the value of $y(100)$ to four decimal places? □

Problem 3. For the equation

$$\frac{du}{dt} = .05u\left(1 - \frac{u}{100}\right)$$

of Problem 1 (b) let u be the solution with $u(0) = 110$.

(a) Use your graphs from Problem 1 (b) to find estimate $u(45)$.
(b) This equation is both separable and a Bernoulli equation. Using the method of your choice find an exact formula for $u(t)$.
(c) What is the value of $u(45)$ to four decimal places. □

Problem 4. A tank starts with 20 liters of distilled water. At time $t = 0$ water that contains 3 grams of salt per liter is pumped into the tank at a rate of .2 liters/minute and well mixed water is pumped out at the same rate. Let

$$w(t) = \text{number of grams of salt in the tank after } t \text{ minutes.}$$

(a) What is the rate that salt is entering the tank. *Hint:* (rate water is entering the tank)×(grams per liter). The units of your answer should be in grams/minute.
(b) At time t there are $w = w(t)$ grams of salt in the tank. How may grams are in liter of water?
(c) At what rate is salt leaving the tanks *Hint:* Rather like part (a) it will be (rate water is leaving the tank)×(grams per liter).
(d) Then

$$\frac{dw}{dt} = (\text{rate salt entering tank}) - (\text{rate salt leaving tank}).$$

Use this to give a differential equation for w.
(e) This equation should be autonomous. Find its critical point(s), sketch the graph of some solutions including the one with $w(0) = 0$, and use this to estimate the amount of salt in the tank after a day (that is $t = 24 \times 60 = 1440$ minutes).
(f) Solve the equation (remembering that $w(0) = 0$) explicitly.
(g) How good was the estimate you gave in in part (b)? □

Problem 5. A cup of tea is 180° F when just poured and 5 minutes later it is 160° F. Assume the room temperature is 65° F and that cup cools according the Newton's Law of cooling: the rate of change of the temperature is proportional to the difference in temperatures. Let $w(t)$ be the temperature of the cup of tea after t minutes. Give a formula for $w(t)$.

If you want more about Newton's law there is a nice exposition at the Kahn Academy: link here, where they derive the general solution using that the equation is separable (I solved it in class by treating it as first order linear). They also give a worked example here. □

Problem 6. A circular disk is losing area at a rate proportional to the length of its boundary. Assume that it starts with a radius of 5 and after 20 minutes it has a radius of 4.5.

(a) Let $r(t)$ be the radius at time t. Find a formula for $r(t)$.
(b) How long until the disk disappears? □