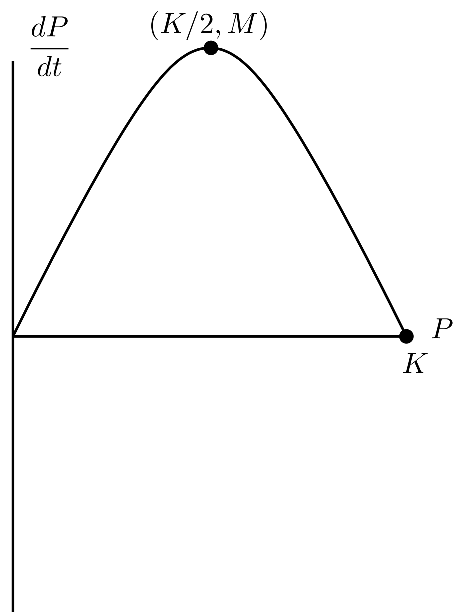
Mathematics 172 Homework, September 30, 2023.

1. Harvesting problems

Here is the basic set up. Suppose we have a population which is at its carrying capacity. To be concrete assume the population grows logistically

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

where, as usual, r is the intrinsic growth rate and K is the carrying capacity. If we plot the rate (i.e. the derivative) $\frac{dP}{dt}$ as a function of P we get the familiar graph

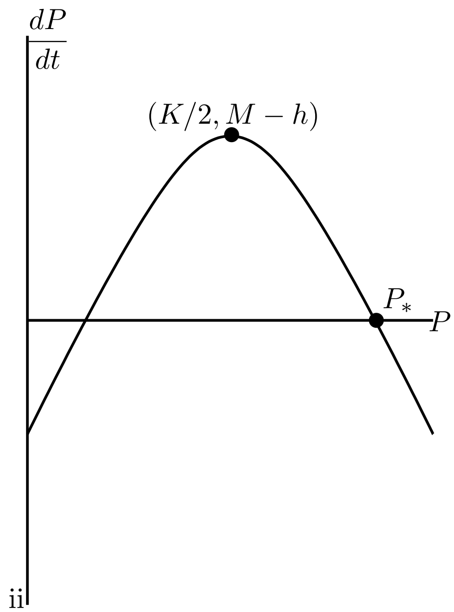


If this figure I have marked the stable equilibrium point K (the other equilibrium point, 0, is unstable). The other marked point is where $\frac{dP}{dt}$ is a maximum. For the logistic equation this occurs when $P = K/2$, that is at half the carrying capacity. In this case the maximum value is $M = r(K/2)^2$. You can also find this point using the graph calculator: 2nd calc 4:maximum.

Now assume that the population is harvested at a constant rate h. Then the rate equation becomes

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h.$$

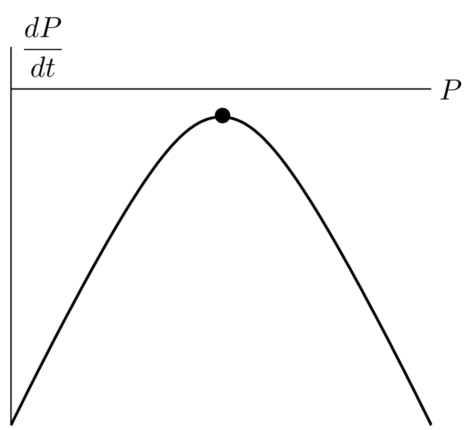
We first assume that h is less than the maximum growth rate of the population. That if $h < M$. Then the graph of $\frac{dP}{dt}$ as a function of P becomes



The graph is just moved down by a distance of h. The point P_* is the stable equilibrium point and can be found using 2nd calc 2:zero.

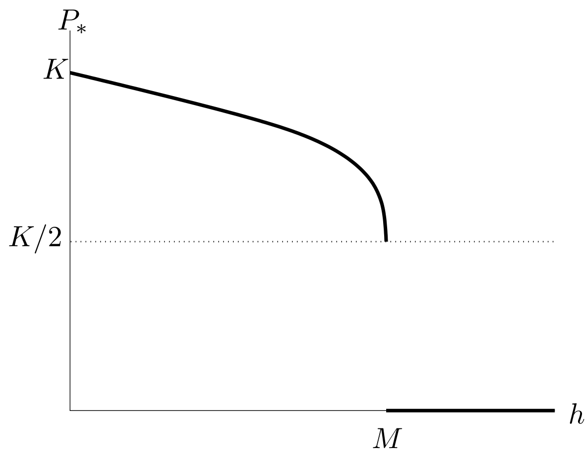
Note that in this case ($h < M$) that the stable population size is P_* is always on the down hill side of the graph (as the graph being decreasing is the condition for stability) and thus the stable population size satisfies $P_* > K/2$. That is when doing harvesting at a constant rate the stable population size is always greater than half the carrying capacity of the ordinal logistic equation.

Finally consider the case where $h > M$. That is the harvesting rate is greater than the maximum possible growth rate of the of the original population. The graph of $\frac{dP}{dt}$ as a function of P now is



This time there are no equilibrium points and $\frac{dP}{dt}$ is always negative. Thus the population dies off.

It is interesting to graph the stable population size P_* as a function of the harvesting rate. The graph looks like



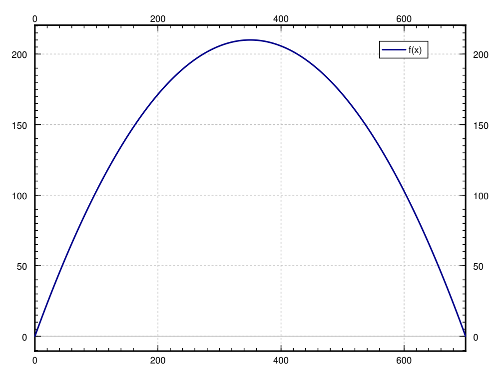
Thus when $h = 0$, the stable population size P_* is just the carrying capacity of the original logistic equation. As h increases the stable population size decreases, but still stays larger than $K/2$ until h becomes larger than M, then the stable population size jumps to zero.

Problem 1. Yeast is being raised in a tank to be sold for home bakers. At first the population of the yeast grows logistically with $r = 1.2$ (lbs/day)lb and a carrying capacity of $K = 700$ lbs. If $A(t)$ is the amount of yeast in the tank after t days, then the rate equation for A is

$$\frac{dA}{dt} = 1.2A\left(1 - \frac{A}{700}\right).$$

(a) What is the maximum rate, h, that the yeast can be harvested with out kill off the entire population?

Solution: The plot of $\dfrac{dA}{dt}$ as a function of A is

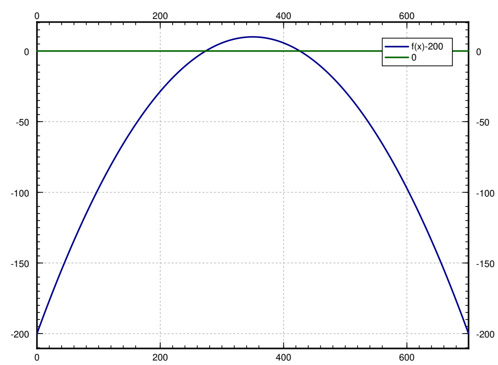


The maximum rate of increase is when $A = 350$ and the maximum value of $\frac{dA}{dt}$ is 210 lbs/day. Therefore the yeast can be harvested at any rate $h < 210$.

(b) If the yeast is harvested at the rate of $h = 200$ lbs/day what is the new stable size of the yeast population. *Solution:* The rate equation is now

$$\frac{dA}{dt} = 1.2A\left(1 - \frac{A}{700}\right) - 200.$$

Again we plot $\frac{dA}{dt}$ as a function of A and this time the graph is



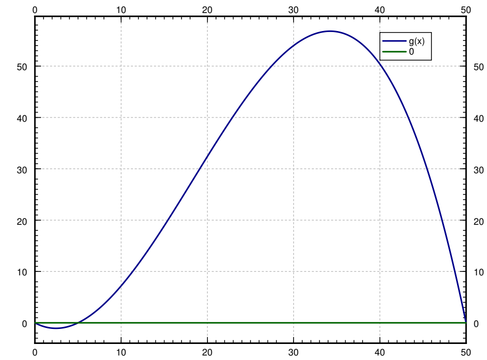
The equilibrium points are where the curve crosses the x-axis. We only want the stable one, which is where the one were the function is decreasing (going downhill). Using the calculator we find this point is $A_* = 426.4$ lbs of yeast.

Problem 2. Here is a problem showing that this method also works for equations other than the logistic equation. Assume we have a population growing by the equation

$$\frac{dA}{dt} = -.9A\left(1 - \frac{A}{5}\right)\left(1 - \frac{A}{50}\right)$$

(a) What is the maximum rate this can be harvested without killing off the population?

Solution: As usual we start by plotting $\frac{dP}{dt}$ as a function. I am using $\mathsf{Xmin} = 0$ and $\mathsf{Xmax} = 50$.



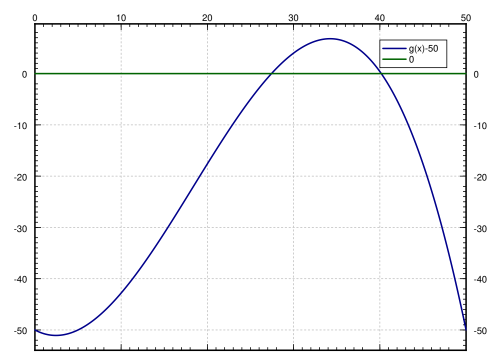
Now use the calculator to find the maximum of $\frac{dA}{dt}$. The result is 56.8. Thus the population can be harvested at any rate $h < 56.8$ without killing off the population.

(b) If the population is harvested at a rate of $h = 50$ lbs/day what is the new stable population size?

Solution: The new rate equation is

$$\frac{dA}{dt} = -.9A\left(1 - \frac{A}{5}\right)\left(1 - \frac{A}{50}\right) - 50.$$

The graph of $\frac{dP}{dt}$ as function of P is



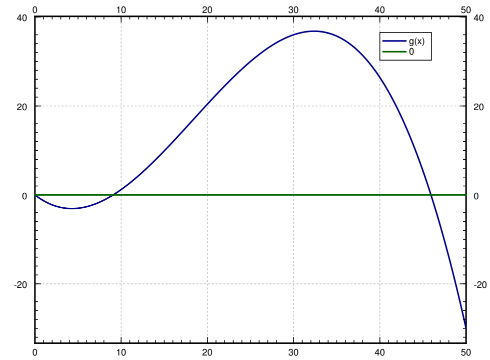
Use the calculator to show the stable equation point is $P_* = 40.17$

(c) Suppose that instead of harvesting at a constant rate, that 60% of the yeast is harvested each day. Find the new stable population size.

Solution: This time the rate equation becomes

$$\frac{dA}{dt} = -.9A\left(1 - \frac{A}{5}\right)\left(1 - \frac{A}{50}\right) - .6P$$

The graph of $\frac{dA}{dt}$ as a function of A is



There are two stable equilibrium points, one of them is $P = 0$, which we ignore, and we can find the other using the calculator and it is $P_* = 45.93$.

Problem 3. Here is a challenge problem for those of you who like algebra. Start with a population growing logistically

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

Let $0 < \rho < r$ and assume that we start harvesting $100\phi\%$. The new rate equation if

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \rho P.$$

Show that this can be rewritten as a logistic equation

$$\frac{dP}{dt} = \tilde{r}P\left(1 - \frac{P}{\tilde{K}}\right).$$

where

$$\tilde{r} = r - \rho, \qquad \text{and} \qquad \tilde{K} = \left(1 - \frac{\rho}{r}\right)K$$