Mathematics 300 Test 2 Name: [redacted]

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1. (15 points) Define the following:

(a) $a \equiv b \pmod{n}$.

For integers a and b and natural number n $a \equiv b \pmod{n}$ if and only if there exists an integer q where $a-b=qn$

(b) a is a ***multiple*** of b.

$b \neq 0$

For integers a and b a is a multiple of b if and only if there exist an integer q where $a=bq$

(c) r is a ***rational number***.

r is a rational ~~number~~ if it can be expressed as $\frac{p}{q}$ $p,q \in \mathbb{Z}$ and $q \neq 0$

(d) α is an ***irrational number***.

α is an irrational number if it cannot be expressed as the quotient of two integers $\alpha \neq \frac{p}{q}$ $p,q \in \mathbb{Z}$ $q \neq 0$

(e) The ***absolute value***, $|b|$, of the number b.

Ther absolute value of b is b if $b \geq 0$ and $-b$ if $b < 0$

$|b| = \begin{cases} b & b \geq 0 \\ -b & b < 0 \end{cases}$

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2. (10 points) (a) Give a precise statement of the ***division algorithm***.

For any integers a and b where $b > 0$ there exists unique integers q and r where $a=qb+r$ and $0 \leq r < b$

(b) What are the quotient and remainder when -18 is divided by 5.

$q =$ -4 $r =$ 2

as $-18 = (-4)5 + 2$

3. (5 points) What is the sum of the geometric series

$$S = 1{,}000 + 1{,}000(1.1) + 1{,}000(1.1)^2 + \cdots + 1{,}000(1.1)^{19}.$$

$\frac{a - ar^{n+1}}{1-r}$ $\frac{1000 - 1000(1.1)^{20}}{1-1.1}$ $= 10{,}000(1.1)^{20} - 10{,}000$

$S =$

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4. (10 points) Prove $n^3 \equiv n \pmod 3$ for all integers n.

proof: assume n is an integer, then there are 3 cases

then there are 3 cases

1: $n \equiv 0 \bmod 3$, $n^3 \equiv 0^3 \bmod 3$

$\equiv 0 \bmod 3$

2. $n \equiv 1 \bmod 3$, $n^3 \equiv (1)^3 \bmod 3$

$\equiv 1 \bmod 3$

3. $n \equiv 2 \bmod 3$, $n^3 \equiv 2^3 \bmod 3$

$\equiv 8 \bmod 3$

$\equiv 2 \bmod 3$

in all cases, $n^3 \equiv n \bmod 3$ for all $n \in \mathbb{Z}$ $\square$

5. (10 points) Prove or give a counterexample: If a^3 is irrational, then a is also irrational.

proof: using the contrapositive, if a is a rational #, then a^3 is a rational #.

$a = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ & $q \neq 0$, then

$a^3 = \left(\frac{p}{q}\right)^3 = \frac{p^3}{q^3} = \frac{A}{B}$ where $A = p^3 \in \mathbb{Z}$ by closure properties & $B = q^3 \in \mathbb{Z}$ and $B \neq 0$ by closure properties,

thus, a^3 is rational when a is rational $\square$

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6. (15 points) Assume the following:

Proposition. *For any integer n, if $5 \mid n^2$, then $5 \mid n$.*

Use this to prove $\sqrt{5}$ is irrational.

PROOF TOWARDS CONTRADICTION:

ASSUME $\sqrt{5}$ IS RATIONAL. BY DEFINITION OF RATIONAL, IT CAN BE EXPRESSED AS $\sqrt{5} = \frac{p}{q}$ WHERE $p, q \in \mathbb{Z}$, $q \neq 0$, & p & q ARE IN LOWEST TERMS

BY ALGEBRA,

$$\sqrt{5}^2 = \left(\frac{p}{q}\right)^2 \Rightarrow 5 = \frac{p^2}{q^2} \Rightarrow 5q^2 = p^2$$

THERE EXISTS AN INTEGER A THAT SATISFIES $p^2 = 5A$, WHERE $A = q^2$. ~~BY ABOVE~~ $5 \mid p^2$ BY THIS FACT & DEFINITION. BY ABOVE PROPOSITION, SINCE $5 \mid p^2$, $5 \mid p$, WE CAN SAY SOME INTEGER g SATISFIES $p = 5g$. WE PLUG THIS BACK INTO $5q^2 = p^2$, & get $5q^2 = (5g)^2$,

$5q^2 = 25g^2 \Rightarrow q^2 = 5g^2$. By THE SAME LOGIC USED FOR p^2, & p, WE CAN CONCLUDE THERE EXISTS AN INTEGER f THAT SATISFIES $q = 5f$, THEREFORE, $\sqrt{5} = \frac{p}{q}$ CAN BE REWRITTEN AS $\frac{5g}{5f}$, WHERE $f \neq 0$. SINCE 5 CAN BE FACTORED OUT OF BOTH p & q, WE HAVE SUCCESSFULLY PROVEN THEY ARE NOT EXPRESSED IN LOWEST TERMS, CONTRADICTING OUR DEFINITION ... DONE

Good

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7. (10 points) Prove directly from the definition of rational number, that if r is rational, then so is $\frac{r-1}{1+r^2}$.

Proof Assume $r \in \mathbb{Q}$, so $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$

So $\frac{r-1}{1+r^2} = \frac{\frac{p}{q} - 1}{1 + \frac{p^2}{q^2}}$ multiply by q^2 $\frac{pq - q^2}{q^2 + p^2}$

$pq - q^2 = m \in \mathbb{Z}$ since integers closed under multiplication and subtraction.

$q^2 + p^2 = n \in \mathbb{Z}$ since integers closed under multiplication and addition.

$\frac{r-1}{1+r^2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$, so $\frac{r-1}{1+r^2} \in \mathbb{Q}$ by definition.

8. (10 points) Prove: If r is rational and θ is irrational, then $\frac{r-\theta}{r+2\theta}$ is irrational. (For this problem you can use that the basic closure properties of $\mathbb{Q}$, that is that $\mathbb{Q}$ is closed under addition, subtraction, multiplication, and division (when defined).)

Proof Assume $r \in \mathbb{Q}$ and $\theta \notin \mathbb{Q}$.

Towards contradiction, assume $\frac{r-\theta}{r+2\theta} = s \in \mathbb{Q}$

$r - \theta = sr + 2s\theta$ $\quad r - sr = 2s\theta + \theta = (2s+1)\theta$

$\frac{r - sr}{2s+1} = \theta$ $\quad \theta \notin \mathbb{Q}$, so $\frac{r - sr}{2s+1} \notin \mathbb{Q}$ Since $r, 2, 1 \in \mathbb{Q}$,

s must not ~~be~~ rational. Because $s = \frac{r-\theta}{r+2\theta}$, $\frac{r-\theta}{r+2\theta}$ is irrational.

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ok, but easier $(a+b)^2 = a^2+2ab+b^2$
$\equiv a^2+b^2 \pmod 2$ because $2 \equiv 0 \pmod 2$

9. (10 points) Prove or give a counterexample: For any integers a and b

$$(a+b)^2 \equiv a^2+b^2 \pmod 2.$$

4 cases

Case 1 - $a \equiv 0$ and $b \equiv 0 \pmod 2$

$(a+b)^2 \equiv (0+0)^2 \pmod 2$
$\equiv 0 \pmod 2$
$a^2+b^2 \equiv 0^2+0^2 \pmod 2$
$\equiv 0 \pmod 2$ ✓

Case 2 - $a \equiv 0$ and $b \equiv 1 \pmod 2$

$(a+b)^2 \equiv (0+1)^2 \pmod 2$
$\equiv 1 \pmod 2$
$a^2+b^2 \equiv 0^2+1^2 \pmod 2$
$\equiv 1 \pmod 2$ ✓

Case 3 - $a \equiv 1$ and $b \equiv 0 \pmod 2$

$(a+b)^2 \equiv (1+0)^2 \pmod 2$
$\equiv 1 \pmod 2$
$a^2+b^2 \equiv 1^2+0^2 \pmod 2$
$\equiv 1 \pmod 2$ ✓

Case 4 - $a \equiv 1$ and $b \equiv 1 \pmod 2$

$(a+b)^2 \equiv (1+1)^2 \pmod 2$
$\equiv 4 \pmod 2$
$\equiv 0 \pmod 2$
$a^2+b^2 \equiv 1^2+1^2 \pmod 2$
$\equiv 2 \pmod 2$
$\equiv 0 \pmod 2$ ✓

In all cases $(a+b)^2 \equiv a^2+b^2 \pmod 2$

10. (5 points) What is the remainder when $N = 123{,}456{,}789$ is divided by 11? *Hint:* $10 \equiv -1 \pmod{11}$.

The remainder is 5

$123456789 = 1(10)^8+2(10)^7+3(10)^6+4(10)^5+5(10)^4+6(10)^3+7(10)^2+8(10)+9$

$\equiv 1(-1)^8+2(-1)^7+3(-1)^6+4(-1)^5+5(-1)^4+6(-1)^3+7(-1)^2+8(-1)+9 \pmod{11}$

$\equiv 1-2+3-4+5-6+7-8+9 \pmod{11}$

-1 2 -2 3 -3 4 -4 5

$\equiv 5 \pmod{11}$

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