Mathematics 554 Homework.

Our immediate goal so to be come experts with inequalities.

The following is an implication what will come up repeatedly.

Proposition 1. *Let $a \in \mathbb{R}$ and $\delta > 0$. Then*

$$|x-a| < \delta \quad \textit{implies} \quad |x| < |a| + 1.$$

Problem 1. Prove this. *Hint:* As you probably ready guessed the first step in this is the adding and subtracting trick:

$$|x| = |x-a+a|$$

and you should be able to use the triangle inequality to finish the proof. □

Recall the reverse triangle inequality:

$$\big||a|-|b|\big| \le |a-b|$$

Note that $|-b| = |b|$ so replacing b by $-b$ in this gives the equivalent inequality

$$|a+b| \ge \big||a|-|b|\big| \ge |a|-|b|.$$

Here is an example of this in action.

Proposition 2. *If $a \ne 0$ and $|x-a| \le \frac{|a|}{2}$, then $|x| \ge \frac{|a|}{2}$.*

Proof. Assume $|x-a| \le \frac{|a|}{2}$. Then we start with the adding and subtracting trick

$$\begin{aligned}
|x| &= |a+(x-a)| && \text{(add and subtract } a\text{)}\\
&\ge |a| - |x-a| && \text{(reverse triangle inequality)}\\
&\ge |a| - \frac{|a|}{2} && \left(\text{as } |x-a| \le \frac{|a|}{2} \text{ so } -|x-a| \ge -\frac{|a|}{2}\ \right)\\
&= \frac{|a|}{2}
\end{aligned}$$

□

Alterantive proof. We have seen that if $|x-c| \le r$ (that is the distance of x from c is at most r) then

$$c - r \le x < c + r. \tag{1}$$

Now assume $|x-a| \le \frac{|a|}{2}$. We first do the case $a > 0$. Then $|x-a| \le \frac{a}{2}$. Using $x = a$ and $r = a/2$ in the inequality (1) then gives

$$a - \frac{a}{2} \le x \le a + \frac{a}{2}.$$

That is

$$\frac{a}{2} \le x \le \frac{3a}{2}$$

which implies $x > 0$ and so we have

$$\frac{a}{2} \leq x = |x|.$$

This leaves the case where $a < 0$. This time use $c = a$ and $r = \frac{|a|}{2} = \frac{-a}{2}$ in (1) to get

$$a - \frac{-a}{2} \leq x \leq a + \frac{-a}{2}$$

which becomes

$$\frac{3a}{2} \leq x \leq \frac{a}{2}.$$

Multiply by -1 and use that $|a| = -a$ and $|x| = -x$ as $a, x < 0$

$$\frac{|a|}{2} \leq \frac{-a}{2} \leq -x = |x| \leq \frac{-3a}{2} = \frac{3|a|}{2}.$$

This covers all cases. □

Problem 2. Use a variant on one of the two the previous proofs to show that if $a \neq 0$ and $|x - a| \leq \frac{1}{5}|a|$, then $|x| \geq \frac{4|a|}{5}$. □

Proposition 3. *Let $a \neq 0$ and let $|x - a| \leq \frac{|a|}{2}$. Then*

$$\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{2}{a^2}|x - a|.$$

Proof. We start with a bit of algebra:

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{a}\right| &= \left|\frac{a - x}{ax}\right| \\ &= \frac{1}{|a||x|}|x - a|. \end{aligned} \tag{2}$$

We saw in Proposition 2 that $|x - a| \leq \frac{|a|}{2}$ implies

$$|x| \geq \frac{|a|}{2}$$

which in turn implies

$$\frac{1}{|x|} \leq \frac{2}{|a|}.$$

Use this in (2) to get

$$\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{2}{|a||a|}|x - a| = \frac{2}{a^2}|x - a|.$$

□

Problem 3. Let $a \neq 0$. Show if $|x-a| \leq \frac{|a|}{5}$, then

$$\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{5}{4a^2}|x-a|.$$

Problem 4. Let $f(x) = 2x^2 + 7x - 9$. Show if $|x-a| \leq 1$, then

$$|f(x) - f(a)| \leq (4|a| + 9)|x-a|. \qquad \square$$

Problem 5. In the notes read section 2.2.1 pages 35–38 and do problems 2.28–2.30, 2.32–2.34.