Mathematics 554 Homework.

This homework is partly about getting examples of Lipschitz functions, but maybe even more important it is practice in using inequalities. Recall that $f\colon [a,b] \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M > 0$ such that

$$|f(x) - f(y)| \le M|x - y|$$

Proposition 1. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lipschitz and let A and B be constants. Then*

$$h(x) = Af(x) + Bg(x)$$

is also Lipschitz.

Problem 1. Prove this. *Hint:* If M_1 is a Lipschitz constant for f and M_2 is a Lipschitz constant for g then you should get $M = M_1|A| + M_2|B|$ for the Lipschitz constant for h. □

Definition 2. Let $f\colon [a,b] \to \mathbb{R}$. Then f is ***bounded*** if and only if there is a constant B such that

$$|f(x)| \le B$$

for all $x \in [a,b]$. □

Proposition 3. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lipschitz and bounded. Then the product*

$$p(x) = f(x)g(x)$$

is also Lipschitz. (More concisely: the product of bounded Lipschitz functions is Lipschitz.)

Problem 2. Prove this. *Hint:* Let B_1 and B_2 be so that

$$|f(x)| \le B_1, \qquad |g(x)| \le B_2$$

on $[a,b]$. These constants exist because f and g are bounded on $[a,b]$. As the functions f and g are Lipschitz there are constants M_1, M_2 such that

$$|f(x) - f(y)| \le M_1|x - y|, \qquad |g(x) - g(y)| \le M_2|x - y|.$$

Now the idea is to use the adding and subtracting trick and write

$$p(x)-p(y) = f(x)g(x)-f(y)g(y) = f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y).$$

You should now be able to use the triangle inequality to show

$$|p(x) - p(y)| \le (B_1M_2 + B_2M_1)|x - y|.$$

□

Proposition 4. *Let $f\colon [a,b] \to \mathbb{R}$ be Lipschitz and assume that there is a constant C such that $|f(x)| \ge C > 0$ on $[a,b]$. Then the function*

$$h(x) = \frac{1}{f(x)}$$

is Lipschitz on $[a,b]$.

Problem 3. Prove this. (No hint this time, this is enough like problems we have already done, so the training wheels are off.)

Actually in some of the above we did not really have to make the extra assumption that the functions are bounded.

Proposition 5. *Let* $f\colon [a,b]\to\mathbb{R}$ *be Lipschitz on the bounded interval* $[a,b]$*. Then* f *is bounded on* $[a,b]$*.*

Problem 4. Prove this. *Hint:* We get to add and subtract yet again. Let $x\in[a,b]$. Then

$$f(x)=f(a)+(f(x)-f(a)).$$

So by the triangle inequality

$$|f(x)|\le|f(a)|+|f(x)-f(a)|.$$

Let M be a Lipschitz constant for f on $[a,b]$. Your should now be able so show

$$|f(x)|\le|f(a)|+M|b-a|$$

which shows f is bounded on $[a,b]$.

Problem 5. Let $f(x)=mx+b$ be a linear function on $\mathbb{R}$. Show

$$|f(x)-f(y)|=|m||x-y|$$

for all $x,y\in\mathbb{R}$. Thus linear functions are Lipschitz on all of $\mathbb{R}$. □

Problem 6. Let $f(x)=Ax^2+Bx+C$ be a quadratic function on the bounded interval $[a,b]$. Then f is Lipschitz on $[a,b]$. *Hint:* We now have enough machinery that we will not have to use any inequalities. We can write $f(x)$ as

$$f(x)=x(Ax+B)+C$$

Then the functions x and $(Ax+B)$ are linear (and thus Lipschitz) on $[a,b]$ and so you can quote a result above as to why they are bounded. Thus $x(Ax+B)$ is product of bounded Lipschitz functions and thus is Lipschitz. Finally the constant function C is Lipschitz thus $f(x)=x(Ax+B)+C$ is the sum of Lipschitz functions and therefore Lipschitz.

Problem 7. Do Problem 2.49 in *Notes on Analysis.* This is the proof of the Lipschitz version of the Intermediate Value Theorem.