Review for test 2.

To begin things in reverse order, here is the most recent material for which you are responsible.

Definition 1. Let $f, f_1, f_2, f_3, \ldots$ be functions on the set A with values in $\mathbf{R}$.

(a) Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

pointwise iff for each $x \in A$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ in the usual sense. That is

$$(\forall x \in A)(\forall \varepsilon > 0)(\exists N)(\forall n)\big[n \geq N \implies |f_n(x) - f(x)| < \varepsilon\big].$$

Note in this case $N = N(x, \varepsilon)$ depends on both x and ε.

(b) And

$$\lim_{n\to\infty} f_n(x) = f(x)$$

uniformly iff for all $\varepsilon > 0$ there is a N that works for all $x \in A$. That is

$$(\forall \varepsilon > 0)(\exists N)(\forall n)(\forall x \in A)\big[n \geq N \implies |f_n(x) - f(x)| < \varepsilon\big].$$

In this case $N = N(\varepsilon)$ only depends on ε and not on x. □

The difference between pointwise and uniform convergence is exactly analogous to the difference between continuity and uniform continuity.

Proposition 2. *Let $f, f_1, f_2, f_3, \ldots$ be functions on the set A with values in $\mathbf{R}$. Assume for some $\varepsilon_0 > 0$ that for each n there is a point x_n there is a $x_n \in A$ such that $|f_n(x) - f(x)| \geq \varepsilon_0$. Then $\langle f_n \rangle_{n=1}^{\infty}$ does not converge uniformly to f.*

Proof. The negation of the definition of the definition of uniform convergence is

$$(\exists \varepsilon > 0)(\forall N)(\exists n)(\exists x \in A)\big[n \geq N \text{ and } |f_n(x)) - f(x)| \geq \varepsilon\big].$$

Let $\varepsilon = \varepsilon_0$ and for any N we choose $n > N$ and choose $x = x_n$. Then $n > N$ and $|f(x) - f_n(x)| = |f(x_n) - f_n(x_n)| \geq \varepsilon_0 = \varepsilon$. Thus the limit is not uniform. □

Example 3. Let $f_n \colon [0,1] \to \mathbf{R}$ be

$$f_n(x) = x^n$$

then, as we have seen in class, we have the pointwise limit

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1; \\ 1, & x = 1. \end{cases}$$

But if $x_n = (1/2)^{1/n}$ we have $|f_n(x_n) - f(x_n)| = |1/2 - 0| = 1/2$ and therefore (using $\varepsilon_0 = 1/2$ in Proposition 2) we see that the limit $\lim_{n\to\infty} f_n = f$ is not uniform. □

Example 4. Let

$$f_n(x) = \frac{nx}{1+n^2x^2}.$$

Then it is not hard to check that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all x. Therefore $\lim_{n\to 0} f_n(x) = 0$ pointwise on $\mathbf{R}$. But $f_n(1/n) = 1/(1+1) = 1/2$. Thus we can again use Proposition 2 with $\varepsilon_0 = 1/2$ to see that the limit $\lim_{n\to\infty} f_n(x) = 0$ is not uniform. □

For the test you should know the definitions above and at least one example to show that pointwise convergence does not imply uniform convergence.

The main topic we have covered since the last test is series. Here is an outline of what you should know

(1) The definition of a series and what it means for it to converge. (Definition 1 from *Notes on Series* from the class web page).

(2) The basic propositions about series (cf. *Notes on Series* Theorem 2, Theorem 3, Proposition 5 and be sure you know how to sum a geometric series.) For some sample problems see the text exercises 4.3 (starting on page 228) Problems 3, 4. Here is an example of a geometric series you should be able to sum: $\sum_{k=n}^{\infty} 5x^{2k+1}$ where $|x| < 1$.

(3) Tests for series of positive (or non-negative) terms to converge. The most basic result is *Notes on Series* Theorem 9, that the series converges if and only if the sequence of partial sums is bounded. In the special case the terms of the series are monotone decreasing this gave the following two tests:
   (a) Cauchy Condensation Test (*Notes on Series* Theorem 11) and
   (b) The Integral Test (*Notes on Series* Theorem 14) (which is the more important of the two.)

   Remember these two tests only apply when the terms are monotone decreasing. An important consequence of these tests is

   Proposition 5. *The series* $\sum_{k=1}^{\infty} 1/k^p$ *converges if and only if* $p > 1$. □

   For problems using these see the text Exercises §4.3, Problem 10.

(4) Comparison tests for series of positive terms. The basic results are *Notes on Series* Proposition 15, Theorem 16 and Corollary 17. You should know both the statements and proofs. Often the limit comparison is used to compare a series with a p-series. For some sample problems see the text Exercises §4.3, Problem 8.

(5) Limit comparisons with a geometric series. These are the root and ratio test (*Notes on Series* Theorem 21 and Theorem 22). For practice in testing series for convergence see the text Exercises §4.3, Problem 15 (not all of these are root or ratio test problems).

(6) Absolute and conditional convergence. Of course know the definitions. A central result here is the ***alternating series test***, that is *Notes on Series* Theorem 28 in *Notes on Series* . Be able to state this theorem and use it to given an example of a conditional convergent series. Another important idea related to absolute and conditional convergence is rearrangements of series. For this see the text, Theorem 4.3.24 and Theorem 4.3.26. In class we only did the version of Theorem 4.3.26 with $\mu = \nu$, that is if a series is conditionally convergent, then the sum can be rearranged to be real number we wish.

(7) Power series. These are series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

We did most of the proofs in the case where $x_0 = 0$. Here is the summary of the main results. The series for $f(x)$ has a ***radius of convergence***, r, which has the general formula

$$r = \frac{1}{\limsup_{n\to\infty} |a^n|^{1/n}}$$

(This is interrupted as $r = 0$ if $\limsup_{n\to\infty} |a^n|^{1/n} = \infty$ and $r = \infty$ if $\limsup_{n\to\infty} |a^n|^{1/n} = 0$.) In most cases r can be more easily found by using either the root or ratio test. (Sample problems on finding r are in the text Exercises §4.5 Problem 2.) The following summarizes several of our theorems.

Theorem 6. *Let the power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

have radius of convergence $r > 0$. Then the series converges absolutely for $|x - x_0| < r$, diverges for $|x - x_0| > r$ (and in general nothing can be said about convergence when $|x - x_0| = r$). For $|x - x_0| < r$ the function $f(x)$ is differentiable and its derivative is given by termwise differentiation of the series for $f(x)$,

$$f'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k$$

and this series has the same radius of convergence. Thus it is also differentiable. Therefore $f(x)$ has derivatives all orders. The coefficients a_k are given in terms of the derivatives of f by

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Finally we can also integrate the series for $f(x)$ termwise in the sense that for $|x - x_0| < r$

$$\int_{x_0}^{x} f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x - x_0)^{k+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k}(x - x_0)^{k}. \qquad \square$$

We used this to to derive or define the following

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

which converge absolute for all $x \in \mathbf{R}$. These you should know from memory. And

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

which converges for $-1 < x \le 1$ and

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $-1 \le x \le 1$. It is not so important that know these by memory, but you should understand the method by which they were derived (i.e. integrating a known series). We also know for $|x| < 1$ that

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

and the statement of Abel's Continuity Theorem. Finally all this stuff on power series is tied up with Taylor's Theorem with remainder, but you are already are experts on that result.

(8) Some miscellaneous problems from the text to try:
   (a) In Exercises §4.3, Problem 12, Problem 15 a,b, Problem 18, Problem 19, Problem 21 a,b,c, Problem 25, b, d, Problem 27 b, c, Problem 33.
   (b) In Exercise §4.5, Problems 10, (*Hint:* $x\frac{x}{dx}x^n = nx^n$.), 16 (*Hint:* $\int_1^x \frac{\ln(t)}{t-1}\,dt = \int_1^x \frac{\ln(1+(t-1))}{t-1}\,dt$ and we know how to expand $\ln(1+x)$ into a series.)