Mathematics 555

Final Name:______________________________

The problems are 25 points each.

1. (a) Let f be defined on a open interval containing the points a. State the definition of f being ***differentiable*** at a.

(b) Assume that f is differentiable at a and that $f(a) \neq 0$. Use the definition of the derivative to show that the function $g(x) = \dfrac{1}{f(x)}$ is differentiable at a and that $g'(a) = \dfrac{-f'(a)}{f(a)^2}$.

2. (a) State ***Rolle's Theorem***.

(b) State the ***Mean Value Theorem***.

(c) Use Rolle's Theorem to prove the Mean Value Theorem.

3. (a) Let $\langle f_n \rangle_{k=1}^{\infty}$ be a sequence of functions on the interval $[a, b]$. Give the definition of

$$\lim_{k\to\infty} f_k(x) = f(x)$$

uniformly on $[a, b]$.

(b) Prove that if each f_k is continuous and $\lim_{k\to\infty} f_k(x) = f(x)$ uniformly on $[a, b]$ then f is also continuous. (That is show that a uniformly convergent sequence of continuous functions has a continuous limit.)

4. Let the function $f(x)$ be defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \frac{x^{14}}{7!} + \cdots$$

(a) What is the interval of convergence of this series?

(b) Show that on the interval of convergence that

$$f'(x) = 2xf(x).$$

5. Give examples of the following:

(a) A sequence of continuous $f_k\colon [0,1]$ such that

$$\lim_{k\to\infty} f_k(x) = 0$$

pointwise but

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = 1.$$

(b) A sequence of continuous $f_k\colon [0,1]$ such that that converges pointwise to a function f, but f is not continuous.

(c) A convergent series that is not absolutely convergent.

(d) A function $f(x, y)$ defined on $\mathbf{R}^2$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x, y)$$

(e) A function that is not Riemannian integrable.

6. For each positive integer n let

$$K_n(x) = \begin{cases} \dfrac{n}{2}, & |x| < \dfrac{1}{n}; \\ \\ 0, & |x| > \dfrac{1}{n}. \end{cases}$$

(a) Graph $y = K_n(x)$.

(b) Show $\displaystyle\int_{-\infty}^{\infty} K_n(y)\,dy = 1$ (you can use your graph and geometry to do this) and use this to show

$$f(x) = \int_{-\infty}^{\infty} f(x) K_n(y)\,dy$$

(c) Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and set

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y) K_n(y)\, dy$$

With the same notation as above show

$$f_n(x) - f(x) = \frac{n}{2} \int_{-1/n}^{1/n} \big(f(x-y) - f(x)\big)\, dy$$

(d) Finally, using that f is continuous, at x show

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Have a good summer!