Mathematics 555 Test 1, Take Home Portion: Answer Key.

1. Let $f\colon (a,b) \to \mathbf{R}$ be twice differentiable with f' and f'' both continuous. Show for $x \in (a,b)$ that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

Solution: Let $h \neq 0$ be so that $x-h, x+h \in (a,b)$ and let $\varepsilon > 0$. By Taylor's Theorem there is ξ_1 between x and $x+h$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{f''(\xi_1)}{2}h^2.$$

Likewise there is ξ_2 between x and $x-h$ with

$$f(x-h) = f(x) - f'(x)h + \frac{f''(\xi_2)}{2}h^2$$

Adding these gives

$$f(x+h) + f(x-h) = 2f(x) + \frac{f''(\xi_1) + f''(\xi_2)}{2}h^2.$$

This can be rearranged to give

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \frac{f''(\xi_1) + f''(\xi_2)}{2}.$$

Now subtract $f''(x)$ from both sides of this and take absolute values to get

$$\begin{aligned}
\left| \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x) \right| &= \left| \frac{f''(\xi_1) + f''(\xi_2)}{2} - f''(x) \right| \\
&= \left| \frac{f''(\xi_1) + f''(\xi_2) - 2f''(x)}{2} \right| \\
&= \left| \frac{(f''(\xi_1) - f''(x)) + (f''(\xi_2) - f''(x))}{2} \right| \\
&\leq \frac{|f''(\xi_1) - f''(x)| + |f''(\xi_2) - f''(x)|}{2}
\end{aligned}$$

As f'' is continuous at x there is a $\delta > 0$ such that

$$|\xi - x| < \delta \quad \text{implies} \quad |f''(\xi) - f''(x)| < \varepsilon.$$

Let $0 < |h| < \delta$. Then as ξ_1 is between x and $x+h$ we have $|x - \xi_1| < \varepsilon$. Likewise ξ_2 is between x and $x-h$ and thus $|x - \xi_2| < \varepsilon$. Therefore if

$0 < |h| < \delta$ we have

$$\begin{aligned}\left|\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - f''(x)\right| &\le \frac{|f''(\xi_1) - f''(x)| + |f''(\xi_2) - f''(x)|}{2} \\ &< \frac{\varepsilon + \varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

which completes the proof. □

2. Let E be a compact metric space and $f, f_1, f_2, f_3, \ldots : E \to \mathbf{R}$ continuous functions. Assume for all $x \in E$ that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and that

$$f_1(x) \ge f_2(x) \ge f_3(x) \ge f_4(x) \ge \cdots$$

(that is the sequence $\langle f_n(x)\rangle_{n=1}^{\infty}$ is monotone decreasing). Show $\lim_{n\to\infty} f_n = f$ uniformly. *Hint:* Let $\varepsilon > 0$ and let

$$U_n = \{x \in E : f_n(x) - f(x) < \varepsilon\}.$$

Quote a theorem from last semester that tells us that U_n is open. Then show

$$U_n \subseteq U_{n+1}$$

and that $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ is an open cover of E.

Solution: As the sequence is monotone decreasing we have

$$f_n(x) - f(x) \ge 0$$

for all x. Let $x \in U_n$. Then

$$\varepsilon > f_n(x) - f(x) \ge f_{n+1}(x) - f(x)$$

and therefore $x \in U_{n+1}$. Thus $U_n \subseteq U_{n+1}$. Also the function $f - f_n$ is continuous and

$$U_n = \{x : (f_n - f)(x) \in (-\infty, \varepsilon)\} = (f_n - f)^{-1}[(-\infty, \varepsilon)].$$

Therefore U_n is the preimage of an open set by a continuous function and thus U_n is open.

Let $x \in E$. Then $\lim_{n\to\infty} f_n(x) = f(x)$ and therefore there is N such that $n \ge N$ implies $|f_n(x) - f(x)| < \varepsilon$. But then

$$f_n(x) - f(x) \le |f_n(x) - f(x)| < \varepsilon.$$

Therefore $x \in U_n$ for $n \ge N$. This shows that $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ is an open cover of E. As E is compact there is there is a finite set

$$\{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\} \subseteq \mathcal{U}$$

that covers. Let

$$N = \max\{n_1, n_2, \dots, n_k\}.$$

Then, as $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$ is a cover of E,

$$E = U_{n_1} \cup U_{n_2} \cup \dots \cup U_{n_k} = U_N.$$

If $n \geq N$, then $U_n \supseteq U_N = E$ and thus for all $x \in E$ if $n \geq N$ we have

$$0 \leq f_n(x) - f(x) < \varepsilon$$

and therefore $|f_n(x) - f(x)| < \varepsilon$. This shows that $\lim_{n\to\infty} f_n = f$ uniformly. □

3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be differentiable at all points and let $a < b$. Assume $f'(a) > 0$ and $f'(b) < 0$. Prove there is c between a and b with $f'(c) = 0$. *Remark:* The derivative f' need not be continuous and therefore this does not follow from the intermediate value theorem. □

Solution: The interval $[a, b]$ is compact and f is continuous. Thus f achieves its maximum on $[a, b]$. We now show that this maximum on $[a, b]$ is at an interior point of the interval. As $f'(a) > 0$ we have

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) > 0.$$

Thus there is a $\delta > 0$ so that if $0 < |x - a| < \delta$, then

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < f'(a).$$

This implies that if $0 < |x - a| < \delta$ then

$$\frac{f(x) - f(a)}{x - a} \geq f'(a) - \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| > 0.$$

Therefore if $a < x < a + \delta$ we have can multiply this by the positive number $x - a)$ to get

$$f(x) > f(a).$$

Thus the maximum does not occur at $x = a$.

Likewise there is a δ_1 such that if $0 < |x - b| < \delta_1$, then (using that $-f(b) > 0$)

$$\left| \frac{f(x) - f(b)}{x - b} - f'(b) \right| < -f'(b).$$

Thus

$$-\frac{f(x) - f(b)}{x - b} \geq -f'(b) - \left| \frac{f(x) - f(b)}{x - b} - f'(b) \right| > 0.$$

Therefore if $b - \delta_1 < x < b$ we can multiply by the positive number $-(x-b)$ to get

$$f(x) > f(b).$$

Thus the maximum of f on $[a,b]$ does not occur at b.

Putting this together gives that the maximum of f occurs at an interior point, c, of $[a,b]$. Thus f has a local maximum at c and thus $f'(c) = 0$. □

4. Let E and E' be a metric space and $f\colon E \to E'$ a function. Let $\alpha > 0$. Then f satisfies a ***Hölder condition*** of order α if and only if there is a constant $C \geq 0$ such that

$$d(f(p), f(q)) \leq C d(p,q)^{\alpha}$$

for all $p, q \in E$.

(a) Show that if f satisfies a Hölder condition, then f is uniformly continuous.

Solution: Let $\varepsilon > 0$ and let

$$\delta = \left(\frac{\varepsilon}{C}\right)^{1/\alpha}.$$

Then if $p, q \in E$ with $d(p,q) < \delta$ we have

$$d(f(p), f(p)) \leq C d(p,q)^{1/\alpha} < C\delta^{\alpha} = C\left(\left(\frac{\varepsilon}{C}\right)^{1/\alpha}\right)^{\alpha} = \varepsilon.$$

Thus f is uniformly continuous. □

(b) Let $f\colon \mathbf{R} \to \mathbf{R}$ satisfy a Hölder condition of order $\alpha > 1$. Show f is constant. (If you want to be a bit more definite it is ok to assume that $\alpha = 2$ in this problem.)

Solution: Let $a \in \mathbf{R}$. Then by the Hölder condition we have for $x \in \mathbf{R}$,

$$|f(x) - f(a)| \leq C|x-a|^{\alpha}.$$

We now use this to show that $f'(a) = 0$. This will show that $f' = 0$ at all points and thus f is constant.

$$\begin{aligned}
\left|\frac{f(x)-f(a)}{x-a}\right| &= \frac{|f(x)-f(a)|}{|x-a|} \\
&\leq \frac{C|x-a|^{\alpha}}{|x-a|} \\
&= C|x-a|^{\alpha-1}.
\end{aligned}$$

Let $\varepsilon > 0$ and set

$$\delta = \left(\frac{\varepsilon}{C}\right)^{1/(\alpha-1)}.$$

Then a calculation as in part (a) shows

$$0 < |x-a| < \delta \quad \text{implies} \quad \left|\frac{f(x)-f(a)}{x-a}\right| < \varepsilon$$

and therefore

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0$$

which completes the proof. □

5. Let $f, f_1, f_2, f_3, \ldots E \to E'$ be maps between metric the metric spaces E and E'. Assume that $\lim_{n\to\infty} f_n = f$ uniformly and that each f_n is uniformly continuous. Show that f is also uniformly continuous. □

Solution: Let $\varepsilon > 0$. As $f_n \to f$ uniformly, there is a N such that

$$n \geq N \quad \text{implies} \quad d(f_n(p), f(p)) < \frac{\varepsilon}{3} \text{ for all } p \in E.$$

Let choose n_0 with $n_0 > N$. The function f_{n_0} is uniformly continuous and thus there is a $\delta > 0$ so that for $p, q \in E$

$$d(p,q) < \delta \quad \text{implies} \quad d(f_{n_0}(p), f_{n_0}(q)) < \frac{\varepsilon}{3}.$$

Whence if $p, q \in E$ with $d(p,q) < \delta$ we have

$$\begin{aligned} d(f(p), f(q)) &\leq d(f(p), f_{n_0}(p)) + d(f_{n_0}(p), f_{n_0}(q)) + d(f_{n_0}(q), f(q)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon, \end{aligned}$$

which shows f is uniformly continuous. □