Mathematics 555 Test 2, Take Home Portion.

The problems are 10 points each. You can choose any four to use as the take home part of the test. If you turn in more than four, I will use the four best four the grade. You are allowed to use any result in the homework notes

1. Let

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Show

$$\lim_{n\to\infty} (H_n - \ln(n))$$

exists. $\square$

Solution. Note we can express $1/k$ as

$$\frac{1}{k} = \int_k^{k+1} \frac{dx}{k}$$

and so we can rewrite $H_n - \ln(n)$ as

$$\begin{aligned}
H_n - \ln(n) &= \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{dx}{x} \\
&= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dx}{x} \\
&= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \int_k^{k+1} \frac{dx}{x} \right) + \frac{1}{n} \\
&= \sum_{k=1}^{n-1} \left(\int_k^{k+1} \frac{1}{k}\, dx - \int_k^{k+1} \frac{dx}{x} \right) + \frac{1}{n} \\
&= \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{1}{k} - \frac{1}{x} \right) dx + \frac{1}{n} \\
&= \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x-k}{kx}\, dx + \frac{1}{n}
\end{aligned}$$

For $k \le x \le k+1$ the inequality

$$\frac{x-k}{xk} \le \frac{(k+1)-k}{(k+1)k} = \frac{1}{k(k+1)} \le \frac{1}{k^2}$$

holds. Thus

$$0 \le \int_k^{k+1} \frac{x-k}{kx}\,dx \le \int_k^{k+1} \frac{1}{k^2}\,dx = \frac{1}{k^2}.$$

Therefore the series

$$\sum_{k=1}^{\infty} \int_k^{k+1} \frac{x-k}{kx}\,dx$$

converges by comparison to the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Combining this with the formula of $H_n - \ln(n)$ just derived gives that

$$\begin{aligned}\lim_{n\to\infty}(H_n - \ln(n)) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\int_k^{k+1}\frac{x-k}{kx}\,dx + \frac{1}{n}\right)\\ &= \sum_{k=1}^{\infty}\int_k^{k+1}\frac{x-k}{kx}\,dx + 0\end{aligned}$$

exists. □

2. Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Assume that $f \ge 0$ on $[a,b]$ and that

$$\int_a^b f(x)\,dx = 0.$$

Prove $f(x) = 0$ for all $x \in [a,b]$. □

Solution 1. Define $F\colon [a,b] \to \mathbf{R}$ by

$$F(x) = \int_a^x f(t)\,dt$$

By the Fundamental Theorem of Calculus

$$F'(x) = f(x) \ge 0.$$

Therefore F is monotone increasing. Thus for $a \le x \le b$

$$F(a) \le F(x) \le F(b) = F(a).$$

Thus F is constant on $[a,b]$ and so

$$f(x) = F'(x) = 0$$

for all $x \in (a,b)$. □

Solution 2. This proof is a bit longer, but in some ways is more natural as it works directly with the definition of continuity. Towards a contradiction assume that there is a $x_0 \in (a,b)$ with $f(x_0) \ne 0$.

Then $f(x_0) > 0$. Let $\varepsilon = f(x_0)/2$. There there is a $\delta > 0$ with $(x_0 - \delta, x_0 + \delta) \subseteq (a, b)$ and such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon = \frac{f(x_0)}{2}.$$

Thus is $|x - x_0| < \delta$ we have

$$f(x) \geq f(x_0) - |f(x) - f(x_0)| > f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2}.$$

Therefore

$$0 = \int_a^b f(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} f(t)\,dt > \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}\,dt = \frac{2\delta f(x_0)}{2} > 0,$$

a contradiction. □

3. Let $\langle a_k \rangle_{k=0}^{\infty}$ be a sequence of real numbers such that $L = \lim_{k\to\infty} a_k$ exists. For $k \geq 1$ set $b_k = (a_{k-1} - a_k)$. Prove (i.e. there should be some ε's and N's)

$$\sum_{k=1}^{\infty} b_k = a_0 - L.$$

□

Solution. Let $B_n = \sum_{k=1}^{n} b_k$ be the n-th partial sum of the series $\sum_{k=1}^{\infty} b_k$. Then this partial sum telescopes:

$$\begin{aligned}
B_n &= \sum_{k=1}^{n} (a_{k-1} - a_k) \\
&= \sum_{k=1}^{n} a_{k-1} - \sum_{k=1}^{n} a_k \\
&= \sum_{k=0}^{n-1} a_k - \sum_{k=1}^{n} a_k \\
&= a_0 - a_n.
\end{aligned}$$

Let $\varepsilon > 0$. Then there is a N such that

$$n \geq N \quad \text{implies} \quad |a_n - L| < \varepsilon.$$

Then if $n \geq N$

$$|B_n - (a_0 - L)| = |(a_0 - a_n) - (a_0 - L)| = |a_n - L| < \varepsilon.$$

Therefore $\lim_{n\to\infty} B_n = a_0 - L$, which is the definition of $\sum_{k=1}^{\infty} b_k = (a_0 - L)$. □

4. Let $p\colon \mathbf{R} \to \mathbf{R}$ be a continuous function and let

$$P(x) = \int_0^x p(t)\,dt.$$

Define a function

$$u(x) = e^{P(x)}.$$

Show that u satisfies $u'(x) = p(x)u(x)$ and $u(0) = 1$. $\square$

Solution. By the Fundamental Theorem of Calculus

$$P'(x) = p(x).$$

Therefore by the chain rule

$$u'(x) = \frac{d}{dx} e^{P(x)} = P'(x)e^{P(x)} = p(x)u(x).$$

Also $P(0) = \int_0^0 p(t)\,dy = 0$ and whence

$$u(0) = e^{P(0)} = e^0 = 1.$$

$\square$

5. Let $f_1, f_2, f_3, \ldots : [a,b] \to \mathbf{R}$ be a sequence of continuous functions and assume that $\lim_{n\to\infty} f_n(x) = f(x)$ and the limit converges uniformly.

(a) Quote a theorem that tells us that f is continuous.

Solution. The uniform limit of continuous function is continuous. $\square$

(b) Prove $\displaystyle\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$ $\square$

Solution. Let $\varepsilon > 0$. Then as $f_n \to f$ uniformly on $[a,b]$ there is N such that

$$n \geq N \quad \text{implies} \quad |f(x) - f_n(x)| < \frac{\varepsilon}{b-a}$$

for all $x \in [a,b]$. Then for $n \geq N$

$$\begin{aligned}
\left| \int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx \right| &= \left| \int_a^b (f(x) - f_n(x))\,dx \right| \\
&\leq \int_a^b |f(x) - f_n(x)|\,dx \\
&< \int_a^b \frac{\varepsilon}{b-a}\,dx \\
&= \varepsilon,
\end{aligned}$$

which is exactly what is needed to show the limit exists and has the required value. $\square$

6. (A generalization of the usual integral test) Let $f\colon [1,\infty) \to \mathbf{R}$ be a twice differentiable function such that

$$\int_1^\infty |f''(x)|\,dx := \lim_{n\to\infty} \int_1^\infty |f''(x)|\,dx < \infty.$$

Show that $\sum_{k=1}^{\infty} f(n)$ converges if and only if $\int_1^\infty f(x)\,dx$ converges.

(Where, by definition, $\int_1^\infty f(x)\,dx$ converging means $\lim_{n\to\infty} \int_0^n f(x)\,dx$ exists.) $\square$