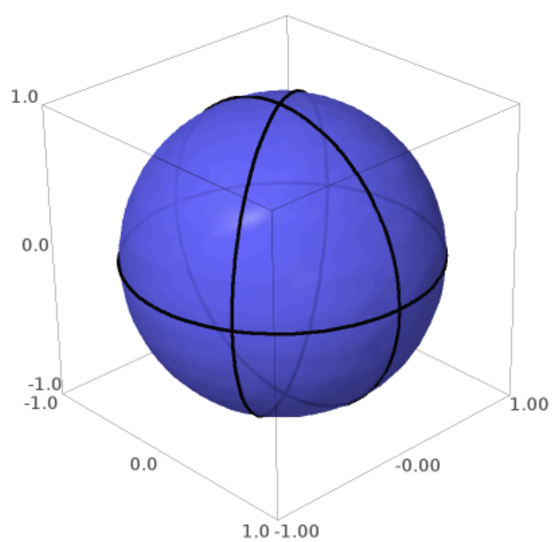
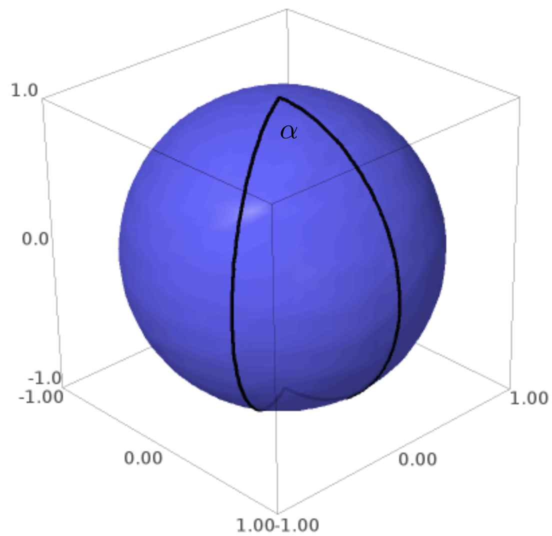
Mathematics 551 Homework, April 1, 2020

In this homework we are going to do some geometry on the sphere that is interesting in its own right, and some of which will generalize to other surfaces. We start with the sphere, S^2, of radius 1 centered at the origin. We know that the area of this is 4π. A ***great circle*** on S^2 is the intersection of S^2 with a plane through the origin. Here is the sphere with three great circles drawn on it.



A segment of a great circle is a ***geodesic segment***. The points p and q on S^2 are ***antipodal*** points if and only if $q = -p$. The basic example of two antipodal points are the north and south poles. A ***lune*** is the region between two geodesic segments connecting a pair of antipodal points. Here is a lune where the angle between the two geodesic segments defining it is α:



Proposition 1. *The area of lune with angle α is*

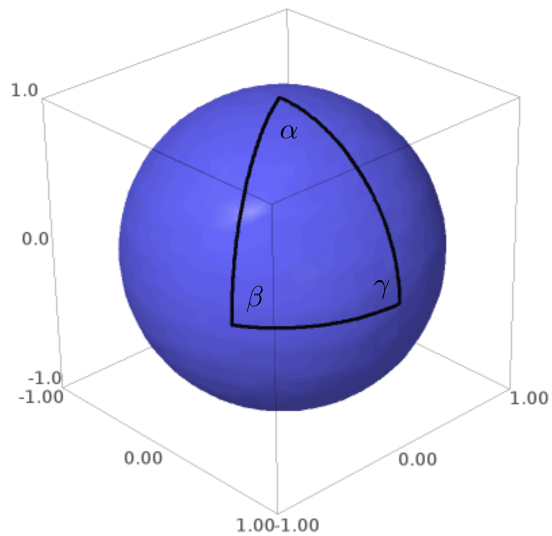
$$A = \frac{\alpha}{2\pi} 4\pi = 2\alpha.$$

Problem 1. As an example of what is going on, note that if $\alpha = \pi/2$, then the lune covers $1/4$ of the sphere is it will have area

$$A = \frac{1}{4} 4\pi = \pi.$$

And in general the proportion of S^2 that is covered by the lune is $\alpha/(2\pi)$. $\square$

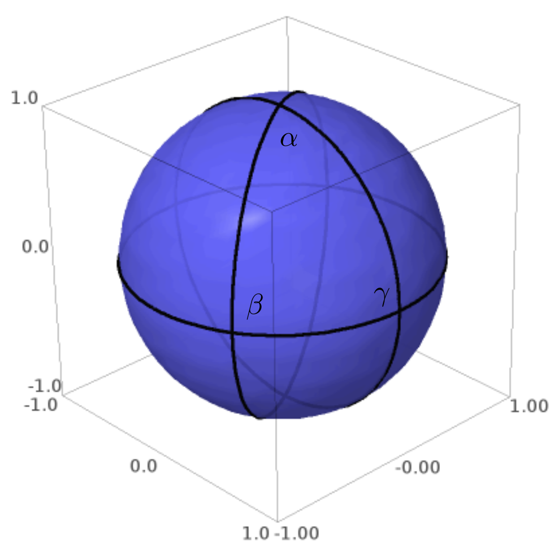
A ***geodesic triangle*** on S^2 is three points connected by the geodesic segments between them like this:



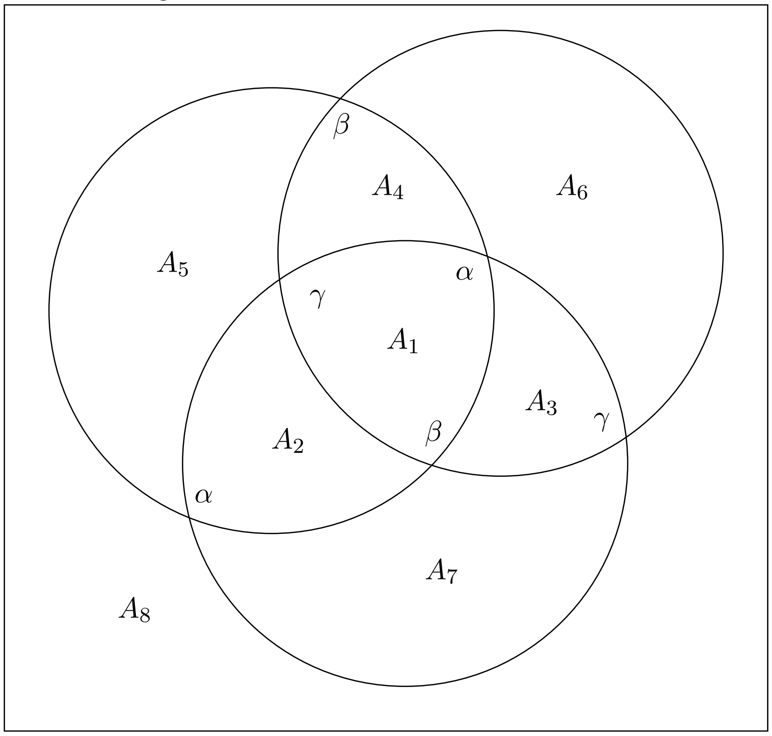
Proposition 2. *With the angles as in the figure the area of the geodesic triangle is*

$$A = \alpha + \beta + \gamma - \pi.$$

Problem 2. Prove this. *Hint:* Expending the sides of the triangle to great circles we get a figure that looks like



These circles split the sphere into 8 triangles which we represent schematically in a Venn diagram



Here the triangle we are interested in is the one with area A_1 at its center, and A_1 is its area (which is what we want to find). Note that $A_1 + A_2$ is the area of a lune with angle α and therefore

$$A_1 + A_2 = 2\alpha.$$

Likewise

$$A_1 + A_4 = 2\beta$$
$$A_1 + A_3 = 2\gamma.$$

Now you should explain that the triangle with A_4 in it is the reflection of the triable with A_7 in it and thus

$$A_4 = A_7.$$

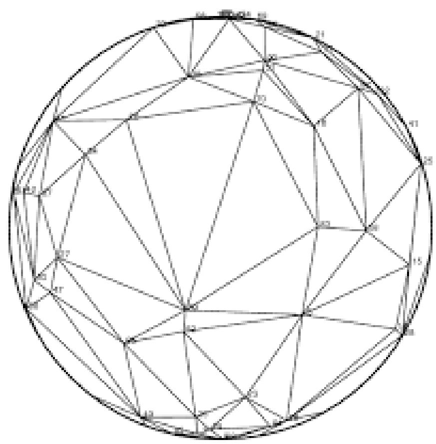
Use similar reasoning to show

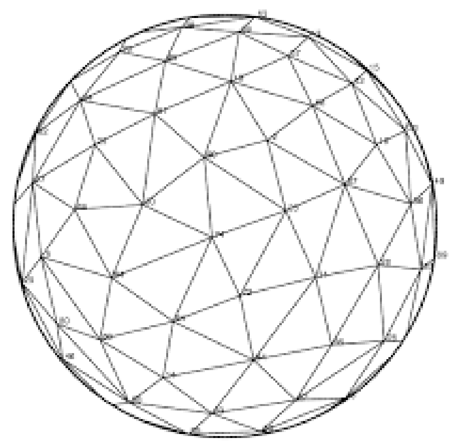
$$A_2 = A_6$$
$$A_3 = A_5$$
$$A_1 = A_8.$$

Finally adding up all the triangles gives the area of the whole sphere and whence

$$\sum_{j=1}^{8} A_j = 4\pi.$$

Now you should be able to solve for A_1. $\square$





Problem 3. By a ***triangulation*** of S^2 we mean splitting it up into geodesic triangles as shown in the two figures above. For a given triangulation let V be the number of vertices, E the number of edges, and F the number of faces (which in this case is the number of triangles). Show that

$$V - E + F = 2.$$

Hint: There are lots of ways to do this, but the one that use the geometry we have been developed here is to let the triangles be $T_1, T_2, \ldots, T_F$ and let α_j, β_j, and γ_j be the angles of T_j and let A_j be the area of T_j. Then the areas add up to 4π and whence

$$\sum_{j=1}^{F} A_j = 4\pi.$$

From the above $A_j = (\alpha_j + \beta_j + \gamma_j) - \pi$ so we have

$$\sum_{j=1}^{F} ((\alpha_j + \beta_j + \gamma_j) - \pi) = 4\pi.$$

Show that this implies the result. (You may have to do a little bit of combinatorics to make this work). $\square$