Mathematics 551 Homework, April 11, 2020

Here we wish to show that if two surfaces have the same first and second fundamental forms then thy differ by a rigid motion. We start with somewhat weaker statement. Let U be a connected open set in the plane and let $\mathbf{f}\colon U \to \mathbb{R}^3$ and $\mathbf{g}\colon U \to \mathbb{R}^3$ be two C^2 maps such that have the same first and second fundamental forms. That is

$$\begin{aligned} \mathbf{f}_u \cdot \mathbf{f}_u &= \mathbf{g}_u \cdot \mathbf{g}_u \\ \mathbf{f}_u \cdot \mathbf{f}_v &= \mathbf{g}_u \cdot \mathbf{g}_v \\ \mathbf{f}_v \cdot \mathbf{f}_v &= \mathbf{g}_v \cdot \mathbf{g}_v \end{aligned}$$

(which is saying they have the same first fundamental forms) and if $\mathbf{n}_{\mathbf{f}}$ and $\mathbf{n}_{\mathbf{g}}$ are the unit normals to $\mathbf{f}$ and $\mathbf{g}$ respectively, then

$$\begin{aligned} \mathbf{f}_{uu} \cdot \mathbf{n}_{\mathbf{f}} &= \mathbf{g}_{uu} \cdot \mathbf{n}_{\mathbf{g}} \\ \mathbf{f}_{uv} \cdot \mathbf{n}_{\mathbf{f}} &= \mathbf{g}_{uv} \cdot \mathbf{n}_{\mathbf{g}} \\ \mathbf{f}_{vv} \cdot \mathbf{n}_{\mathbf{f}} &= \mathbf{g}_{vv} \cdot \mathbf{n}_{\mathbf{g}} \end{aligned}$$

which is the same as saying the second fundamental forms are the same.

Recall that for two plane curves $\mathbf{c}_1, \mathbf{c}_2$ with the same curvature $\kappa_1 = \kappa_2 = \kappa$ we let the unit tangents and unit normals to $\mathbf{c}_1$ and $\mathbf{c}_2$ be $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{n}_1$, and $\mathbf{n}_2$ and then the main step in showing that $\mathbf{c}_1$ and $\mathbf{c}_2$ differed by a rigid motion was to define the function

$$h = \|\mathbf{t}_1 - \mathbf{t}_2\|^2 + \|\mathbf{n}_1 - \mathbf{n}_2\|^2$$

and using the Frenet formulas to show

$$\frac{d}{ds} h = 0$$

and therefore that h is constant.

An extension of this idea works for surfaces. This time let

$$h = \|\mathbf{f}_u - \mathbf{g}_u\|^2 + \|\mathbf{f}_v - \mathbf{g}_v\|^2 + \|\mathbf{n}_{\mathbf{f}} - \mathbf{n}_{\mathbf{g}}\|^2.$$

Then a calculation that requires both ingenuity and tenancy show that the partial derivatives h_u and h_v are both zero and therefore h is constant. For details see the proof of Theorem 3.8 in Shifrin's book.