Mathematics 551 Homework, April 19, 2020

A surface $M \subset \mathbb{R}^3$ is ***ruled*** if each point $p \in M$ is contained in a line, ℓ of $\mathbb{R}^3$ that which is contained in M. That is

$$p \in \ell \subset M.$$

The lines are called the ***rulings*** The easiest examples of ruled surfaces are ***cylinders***. These are constructed by starting with a plane curve

$$\mathbf{c}(s) = (x(s), y(s))$$

and extending by lines parallel to the z-axis. That is

$$\mathbf{x}(u, v) = (x(u), y(u), v).$$

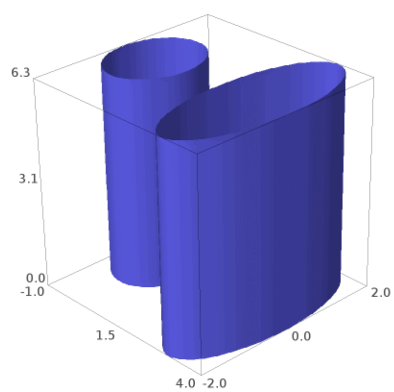

FIGURE 1. The cylinders over the circle $x^2 + y^2 = 1$ and $(x-3)^2 + y^2/2^2 = 1$.

Problem 1. Assume that $\mathbf{c}$ is a unit speed. Show that the first fundamental form of the cylinder is

$$I = du^2 + dv^2$$

and that if κ is the curvature of $\mathbf{c}$ as a plane curve then the principle curvatures of M are $k_1 = \kappa$ and $k_1 = 0$. □

But most ruled surfaces are not cylinders. One example is the helicoid which is the curve

$$\mathbf{x}(u, v) = (v\cos(u), v\sin(u), v) = vE_1(u) + vE_3$$

with our usual notation

$$E_1(u) = (\cos(u), \sin(u), 0), \quad E_2(u) = (-\sin(u), \cos(v), 0), \quad E_3 = (0, 0, 1).$$

Anther interesting example is the ***hyperboloid of one sheet*** defined by the equation

$$x^2 + y^2 - z^2 = 1.$$

See Figure 4. At first glance it may not look like this surface contains any lines. But in fact is is ***doubly ruled***. That is through any point of the surface there are two lines in the surface through the point. Figure 5

Problem 2. For each real number θ define two lines

$$\begin{aligned}\boldsymbol{\alpha}(t) &= (\cos(\theta) - t\sin(\theta), \sin(\theta) + t\cos(t), t),\\ \boldsymbol{\beta}(t) &= (\cos(\theta) + t\sin(\theta), \sin(\theta) - t\cos(t), t).\end{aligned}$$

Show that these two line both are contained in the hyperboloid $x^2+y^2-z^2 = 1$. □

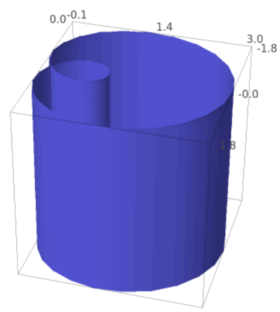

FIGURE 2. The cylinder over the curve with polar equation $r = 1 + 2\cos(\theta)$.

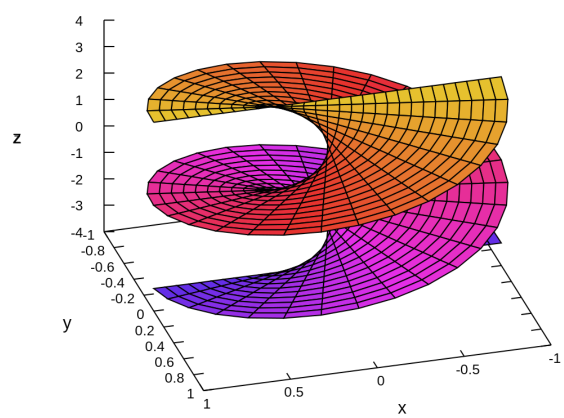

FIGURE 3. The helicoid. The rulings are lines perpendicular to the z-axis.

Back to the general theory.

Proposition 1. *If M is a rules surface, then $K \leq 0$ (where K is the Gauss curvature).*

Problem 3. Prove this. □

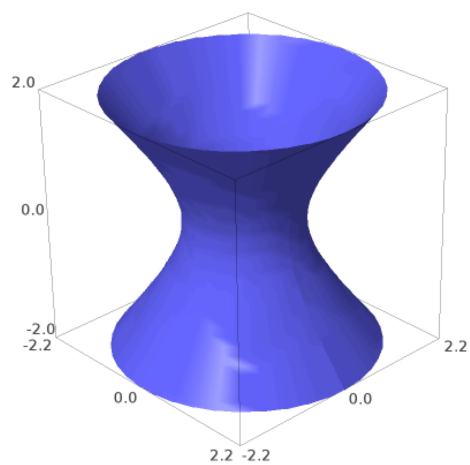

FIGURE 4. The hyperboloid $x^2 + y^2 - z^2 = 1$

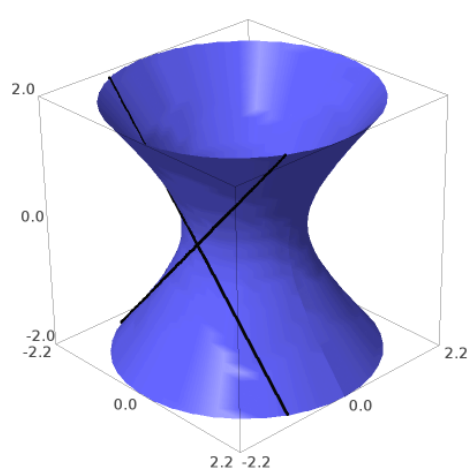

FIGURE 5. The hyperboloid $x^2 + y^2 - z^2 = 1$ showing two lines on the surface through a point.

We now look closer at the structure of ruled surfaces. Let M be ruled and along choose a curve $\mathbf{c}(u)$ with $a < u < b$ in M that intersects the rulings and is not tangent to any of them. Call such a curve ***transverse to the rulings***. Let $\mathbf{r}(u)$ be a unit vector field along $\mathbf{c}(u)$ so that $\mathbf{r}(s)$ is tangent to the ruling. See Figure 6. With this set up we have that the surface is parameterized by

$$\mathbf{x}(u,v) = \mathbf{c}(u) + v\mathbf{r}(u), \qquad a < u < b, \qquad -\infty < v < \infty.$$

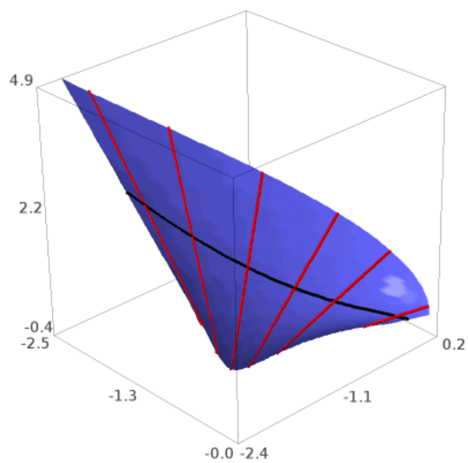

FIGURE 6. A ruled surface (the rulings are shown in red) and a curve $\mathbf{c}$ (shown in black) that interests all the rulings. At each point $\mathbf{c}(u)$ of $\mathbf{c}$ let $\mathbf{r}(u)$ be a unit vector that points along the ruling.

Problem 4. Let $h(u)$ be a smooth real valued function defined for $a < u < b$ and let

$$\mathbf{f}(u,v) = \mathbf{c}(u) + (h(u) + v)\mathbf{r}(u) \qquad a < u < b, \qquad -\infty < v < \infty.$$

also parameterizes M. □

Phrased a bit differently let

$$\boldsymbol{\gamma}(u) = \mathbf{c}(u) + h(u)\mathbf{r}(u), \qquad a < u < b.$$

Then $\boldsymbol{\gamma}$ is also transverse to the rulings and

$$\mathbf{f}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{r}(u)$$

is a parameterization of M.

Since we can choose h is infinitely many ways there are infinitely many choices for the curve $\mathbf{c}$ transverse to the rulings. We would like to make a choice of a particularly nice one.

To avoid some degenerate cases we make the assumption that

$$\mathbf{r}'(u) \neq \mathbf{0}$$

at any point. And we are assuming that $\mathbf{r}$ is a unit vector field and therefore $\|\mathbf{r}(u)\| = 1$. This implies

$$\mathbf{r}(u) \cdot \mathbf{r}'(u) = 0.$$

Problem 5. With our set up here let

$$\boldsymbol{\gamma}(u) = \mathbf{c}(u) + h(u)\mathbf{r}(u)$$

where

$$h(u) = -\frac{\mathbf{c}'(u) \cdot \mathbf{r}'(u)}{\mathbf{r}'(u) \cdot \mathbf{r}'(u)}$$

then $\boldsymbol{\gamma}(u)$ satisfies

$$\boldsymbol{\gamma}'(u) \cdot \mathbf{r}'(u) = 0.$$

This curve is the ***curve of striation*** of the surface. □

Problem 6. On the helicoid show that if the z-axis is parameterized by

$$\boldsymbol{\gamma}(u) = (0, 0, u)$$

then the vector

$$\mathbf{r}(u) = E_1(u) = (\cos u, \sin u, 0)$$

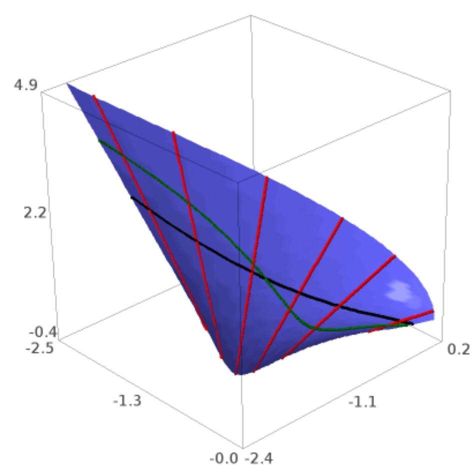

FIGURE 7. The curve in green is a curve of the form $\boldsymbol{\gamma}(u) = \mathbf{c}(u) + h(u)\mathbf{r}(u)$.

is a unit vector field tangent to rulings. Use this to check that

$$\boldsymbol{\gamma}'(u) \cdot \mathbf{r}'(u) = 0$$

and thus $\boldsymbol{\gamma}$ is the curve of striation on the helicoid. □

Problem 7. On the surface $x^2 + y^2 - z^2 = 1$ the intersection with the xy-plane is the circle parameterized by

$$\boldsymbol{\gamma}(u) = (\cos(u), \sin(u), 0)$$

Show that the vector field

$$\mathbf{r}(u) = \frac{1}{\sqrt{2}} (-\sin(u), \cos u, 1)$$

is a unit vector tangent to one set of rulings and that

$$\boldsymbol{\gamma}'(u) \cdot \mathbf{r}'(u) = 0$$

and therefore $\boldsymbol{\gamma}$ is the curve of striation of this hyperboloid. □