Mathematics 552 Homework, January 23, 2020

Problem 1. Compute the following:

(a) The product $(z^2 + (3 + i)z - 9)(2z^2 + 4iz - 9)$.
(b) The quotient and remainder when $z - 1 + 2i$ is divided into $z^3 - 3iz^2 + 4z + (2 + 3i)$.
(c) The roots of $z^2 + (2 + 2i)z + 1 + 2i = 0$. *Hint:* One way is to use the quadratic formula. □

Problem 2. Recall that we have shown that if $p(z)$ is a polynomial of degree n and that if α is a root of $p(z) = 0$, then $p(z)$ factors as $p(z) = (z - \alpha)q(z)$ where $q(z)$ has degree $(n - 1)$. Use this and induction to show that a polynomial of degree n has at most n roots. □

Problem 3. Find all the fourth roots of $16i$ draw a picture showing them. □

1. The binomial theorem

1.1. **Factorials and binomial coefficients.** We recall the definition of the ***factorials***. If n is a non-negative integer $n!$ is defined by

$$0! = 1 \quad \text{and for } n \geq 1 \quad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For small values of n we have

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880

n	$n!$
10	3,628,800
11	39,916,800
12	47,9001,600
13	622,7020,800
14	87,178,291,200
15	1,307,674,368,000
16	20,922,789,888,000
17	3556,87,428,096,000
18	6,402,373,705,728,000
19	121,645,100,408,832,000

n	$n!$
20	2,432,902,008,176,640,000
21	51,090,942,171,709,440,000
22	1,124,000,727,777,607,680,000
23	25,852,016,738,884,976,640,000
24	620,448,401,733,239,439,360,000
25	15,511,210,043,330,985,984,000,000
26	403,291,461,126,605,635,584,000,000
27	10,888,869,450,418,352,160,768,000,000
28	304,888,344,611,713,860,501,504,000,000
29	8,841,761,993,739,701,954,543,616,000,000
30	265,252,859,812,191,058,636,308,480,000,000

Remark 1. These tables make it clear that $n!$ grows very fast. There is a well known approximation, ***Stirling's formula***,

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

which shows that $n!$ grows faster than any exponential function. A more precise form of this was given by Herbert Robbins in 1955:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}.$$

for all positive integers n. □

An elementary property of factorials we will use many times is that we get $n!$ by multiplying $(n-1)!$ by n. Thus

$$\begin{aligned} n! &= n((n-1)!) \\ &= n(n-1)((n-2)!) \\ &= n(n-1)(n-2)((n-3)!) \end{aligned}$$

and so on. This especially useful when dealing with fractions involving factorials. For example:

$$\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)n((n-1)!)} = \frac{1}{(n+2)(n+1)n}.$$

Let $n, k \geq 0$ be integers with $0 \leq k \leq n$. Then the ***binomial coefficient*** $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

This is read as "n choose k".

Problem 4. Show this this definition implies

$$\binom{n}{k} = \binom{n}{n-k}.$$

□

Also we generally do not have to compute $n!$ to find $\binom{n}{k}$ as lots of terms cancel. For example

$$\binom{100}{3} = \frac{100!}{3!\cdot 97!} = \frac{100\cdot 99\cdot 98\cdot 97!}{3!\cdot 97!} = \frac{100\cdot 99\cdot 98}{3!} = 161{,}700.$$

Proposition 2. *The following hold*

$$\begin{aligned}\binom{n}{0} &= \binom{n}{n} = 1,\\ \binom{n}{1} &= \binom{n}{n-1} = n,\\ \binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2},\\ \binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.\end{aligned}$$

Problem 5. Prove this. □

More generally we have

Proposition 3. *The equality*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

holds. □

Here is anther basic property of the binomial coefficients.

Proposition 4 (Pascal Identity)**.** *For $1 \le k \le n$ with k, n integers the equality*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

holds.

Problem 6. Prove this. *Hint:* Here is a special case

$$\begin{aligned}\binom{12}{7} + \binom{12}{8} &= \frac{12!}{7!\,5!} + \frac{12!}{8!\,4!}\\ &= \frac{12!}{7!\,4!}\left(\frac{1}{5}+\frac{1}{8}\right)\\ &= \frac{12!}{7!\,4!}\left(\frac{8+5}{5\cdot 8}\right)\\ &= \frac{12!}{7!\,4!}\left(\frac{13}{5\cdot 8}\right)\\ &= \frac{13!}{8!\,5!}\\ &= \binom{13}{8}\end{aligned}$$

where we have used $13! = 12! \cdot 13$, $8! = 7! \cdot 8$, and $5! = 4! \cdot 5$. □

If we put the binomial coefficients in a triangular table (***Pascal's triangle***):

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ tells us that any entry is the sum of the two entries directly above. This can be used to compute $\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given by:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

1.2. **The binomial theorem.** One reason the binomial coefficients are important is

Theorem 5 (Binormal Theorem)**.** *For any positive integer n and $x, y \in \mathbb{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We will prove this shortly. For $n = 5$ we have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Let $x = y = 1$ in this to get

$$
\begin{aligned}
2^5 &= (1+1)^5 \\
&= (1)^5 + 5(1)^4(1) + 10(1)^3(1)^2 + 10(1)^2(1)^3 + 5(1)(1)^4 + (1)^5 \\
&= 1 + 5 + 10 + 10 + 5 + 1,
\end{aligned}
$$

which may not be that interesting of a fact, but the argument lets us see a pattern for something that is interesting.

Problem 7. Use this idea to show the sum of the numbers $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is 2^n. That is for all positive integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Problem 8. Prove for any positive integer n that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: $(1 - 1) = 0$. □

Here is a bit of practice in using the binomial theorem.

Problem 9. Expand the following:

(a) $(1 + 2x^3)^4$,
(b) $(x^2 - y^5)^3$.

Problem 10. Use induction and the Pascal Identity to prove the Binomial Theorem. *Hint:* Use for the base case that $(x + y)^1 = x + y$. Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}y^4 \\ &= x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\end{aligned}$$

where we have used that $\binom{4}{0} = \binom{4}{4} = 1$. We now want to show the theorem holds for $n = 5$.

$$
\begin{aligned}
(x+y^5) &= (x+y)(x+y)^4 \\
&= (x+y)\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&\quad + y\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + xy^4 \\
&\quad + x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}x^1y^4 + y^5 \\
&= x^5 + \left(\binom{4}{0} + \binom{4}{1}\right)x^4y + \left(\binom{4}{1} + \binom{4}{2}\right)x^3y^2 \\
&\quad + \left(\binom{4}{2} + \binom{4}{3}\right)x^2y^3 + \left(\binom{4}{3} + \binom{4}{4}\right)xy^4 + y^5 \\
&= x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5
\end{aligned}
$$

If you don't like this long hand what of doing it, here is what the same calculation looks like using summation notation. Assume that

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}.$$

Then

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= x(x+y)^4 + y(x+y)^4 \\
&= x\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} + y\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} \\
&= \sum_{k=0}^{4}\binom{4}{k}x^{k+1}y^{4-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^k y^{4-(k-1)} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^k y^{5-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \binom{4}{4}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1} + \binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1} + \binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\binom{5}{k}x^k y^{5-k} \\
&= \sum_{k=0}^{5}\binom{5}{k}x^k y^{5-k}.
\end{aligned}$$

where we have done the change of variable $k \mapsto k-1$ in the first sum on line 5, used the Pascal Identity to get to the second to the last line, and used that $\binom{4}{0} = \binom{5}{0} = 1$ and $\binom{4}{4} = \binom{5}{5} = 1$.

Either of these two calculations shows that if the Binomial Theorem holds for $n = 4$ then it holds for $n = 5$. Use a similar calculation to show that if the theorem holds for n, then in holds for $n+1$. □

Here is an example of one (of the many) ways we will be using the binomial theorem. Similar to some examples given above we will want to simplify expressions of the form

$$\frac{f(z+h) - f(z)}{h}$$

by cancelling the h out of the denominator so that we can compute the limit

$$\lim_{h\to 0}\frac{f(z+h) - f(z)}{h}.$$

Here is an example. Let $f(z) = z^4$. Then

$$\begin{aligned}\frac{f(z+a)-f(z)}{h} &= \frac{(z+h)^4 - z^4}{h} \\ &= \frac{z^4 + 4z^3h + 6z^2h^2 + 4zh^3 + h^4 - z^4}{h} \\ &= \frac{h(4z^3 + 6z^2h + 4zh^2 + h^3)}{h} \\ &= 4z^3 + 6z^2h + 4zh^2 + h^3.\end{aligned}$$

Whence

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}(4z^3 + 6z^2h + 4zh^2 + h^3) = 4z^3.$$

Problem 11. Let $f(z) = z^3$. Compute

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}. \qquad \square$$