Math 552 Test 1, Answer Key

1. *Compute the following and put the answer in the form $x + iy$.*

(a) $(a + bi)^5$ Hint: *Binomial Theorem.*

Solution.

$$\begin{aligned}(a+ib)^5 &= a^5 + 5a^4(bi) + 10a^3(bi)^2 + 10a^2(bi)^3 + 5a(bi)^4 + (bi)^5 \\ &= (a^5 - 10a^3b^2 + 5ab^4) + (5a^4b - 10a^2b^3 + b^5)i\end{aligned}$$

□

(b) $\displaystyle\sum_{k=0}^{11} 3(1+i)^k$

Solution. This is a geometric series. The sum is

$$\begin{aligned}S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3(1+i)^0 - 3(1+i)^{12}}{1 - (1+i)}\end{aligned}$$

Note $(1+i)^0 = 1$ and

$$(1+i)^{12} = \left((1+i)^2\right)^6 = (2i)^6 = -64.$$

Thus

$$S = \frac{3 - 3(-64)}{-i} = \frac{195}{-i} = 195i.$$

□

(c) The solution to $\dfrac{z-i}{z+i} = 2 + 3i$.

Solution. Since I find doing algebra with variables easier than with numbers I am letting $a = 2 + 3i$. Then we are solving

$$\frac{z-i}{z+i} = a.$$

Cross multiply to get

$$z - i = a(z+i) = az + ia.$$

Bring the z terms to the left and the constant terms to the right to get

$$z - az = i + ia$$

Factoring

$$(1-a)z = i(1+a).$$

Thus gives

$$\begin{aligned} z &= \frac{i(1+a)}{1-a)} \\ &= \frac{i(1+2+3i)}{1-(2+3i)} \\ &= \frac{-3+3i}{-1-3i} \\ &= \frac{(-3+3i)(-1+3i)}{(-1-3i)(-1+3i)} \\ &= \frac{-6-12i}{(-1)^2+3^2} \\ &= \frac{-3}{5} + \frac{-6}{5}i \end{aligned}$$

as the solution. □

2. *Let $a = 1 - i$ and let $n = 4k$ where k is an integer. Show a^n is a real number.*

Solution. Write a in polar form

$$a = \sqrt{2}e^{-\pi i/4} = 2^{\frac{1}{2}}e^{-\pi i/4}.$$

Then for $n = 4k$

$$a^n = \left(2^{\frac{1}{2}}e^{-\pi i/4}\right)^{4k} = 2^{2k}e^{-k\pi i} = 4^k\left(e^{-\pi i}\right)^k = 4^k(-1)^k$$

which is a real number. □

3. *Let $p(z)$ be the polynomial*

$$p(z) = c_4z^4 + c_3z^3 + c_2z^2 + c_1z + c_0$$

where the coefficients $c_0 \ldots c_4$ are real numbers.

(a) *Show for any complex numbers z that*

$$\overline{p(z)} = p(\overline{z}).$$

Solution. Since the c_j's are real numbers $\overline{c_j} = c_j$. Now just compute

$$\begin{aligned}
\overline{p(z)} &= \overline{c_4 z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0} \\
&= \overline{c_4 z^4} + \overline{c_3 z^3} + \overline{c_2 z^2} + \overline{c_1 z} + \overline{c_0} \\
&= \overline{c_4}\,\overline{z^4} + \overline{c_3}\,\overline{z^3} + \overline{c_2}\,\overline{z^2} + \overline{c_1}\,\overline{z} + \overline{c_0} \\
&= c_4 \overline{z}^4 + c_3 \overline{z}^3 + c_2 \overline{z}^2 + c_1 \overline{z} + c_0 \\
&= p(\overline{z}).
\end{aligned}$$

where we have used our standard formulas for the complex conjugate including $\overline{z}^k = \overline{z^k}$. □

(b) *Use this to show that if α is a root of $p(z)$, then so is $\overline{\alpha}$. (These facts are true for polynomials with real coefficients of any degree, but you only need to do the case of degree $= 4$.)*

Solution. If α is a root of $p(z)$, then $p(\alpha) = 0$, then using part (a) of this problem we have

$$p(\overline{\alpha}) = \overline{p(\alpha)} = \overline{0} = 0.$$

Thus $\overline{\alpha}$ is also a root. □

4.

(a) *If w is a complex number with $\overline{w} = \dfrac{1}{w}$, then $|w| = 1$.*

Solution. If $\overline{w} = \dfrac{1}{w}$, then

$$|w|^2 = w\overline{w} = w\left(\frac{1}{w}\right) = 1.$$

Taking the (positive) square root gives $|w| = 1$. □

(b) *Show that for any real number x the complex number $w = \dfrac{x+i}{x-i}$ satisfies $|w| = 1$.*

Solution 1. Since x is real, $\overline{x} = x$ and therefore $\overline{x+i} = x - i$ and $\overline{x-i} = x + i$. Whence

$$\overline{w} = \overline{\left(\frac{x+i}{x-i}\right)} = \frac{\overline{x+i}}{\overline{x-i}} = \frac{x-i}{x+i} = \frac{1}{w}.$$

So by part (a) this implies $|w| = 1$. □

Solution 2. Let $z_1 = x + i$ and $z_2 = x - i$. Then

$$|z_1| = \sqrt{x^2 + 1^2} = \sqrt{x^2 + 1}, \qquad |z_2| = \sqrt{x^2 + (-1)^2} = \sqrt{x^2 + 1}.$$

Therefore $|z_1| = |z_2|$. Whence

$$|w| = \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = 1.$$

□

5. *Find both values of* $\sqrt{-2 + 2i\sqrt{3}}$ *in the form* $x + iy$.

Solution. A polar form of $-2 + 2i$ is

$$-2 + 2i\sqrt{3} = 4e^{\frac{2\pi i}{3}}.$$

At square roots are given by

$$\left(-2 + 2i\sqrt{3}\right)^{\frac{1}{2}} = \left(4e^{\frac{2\pi i}{3} + 2n\pi i}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} e^{\frac{\pi i}{3}} \left(e^{i\pi}\right)^n = 2\left(\frac{1 + \sqrt{3}i}{2}\right)(-1)^n = (-1)^n(1 + \sqrt{3})$$

for n an integer. Thus the two square roots of $-2 + 2i\sqrt{3}$ are

$$1 + \sqrt{3}i \qquad \text{and} \qquad -1 - \sqrt{3}i.$$

□

6.

(a) *Find all solutions to* $e^{3z+2} = 1 - i$.

Solution. Writing $1 - i$ in polar form we have

$$e^{3z+2} = \sqrt{2}e^{-\frac{\pi}{4}i} = \sqrt{2}e^{-\frac{\pi}{4}i + 2\pi ni}$$

Therefore

$$3z + 2 = \ln(\sqrt{2}) - \frac{\pi}{4}i + 2\pi ni.$$

Solving this for v gives

$$z = \frac{1}{3}\left(\ln(\sqrt{2}) - 2\right) - \frac{\pi}{6}i + \frac{2}{3}\pi n i.$$

as n varies over the integers. As $\frac{11\pi}{6} = -\frac{\pi}{6} + 2\pi$, this is equivalent to to

$$z = \frac{1}{3}\left(\ln(\sqrt{2}) - 2\right) + \frac{11\pi}{6}i + \frac{2}{3}\pi n i$$

with $n \in \mathbb{Z}$. □

(b) *Let a be a real number with $a > 1$. Find all solutions to $\tan(z) = ia$.*

Solution. Using the definition of $\tan(z)$ in terms of e^{-z} the equation becomes

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = ia$$

Clearing of fractions and using $i^2 = -1$ gives

$$e^{iz} - e^{-iz} = -a(e^{iz} + e^{-iz})$$

Multiple by e^{iz}

$$(e^{iz})^2 - 1 = -a\left((e^{iz})^2 + 1\right)$$

Solving for $\left(e^{iz}\right)^2$ gives

$$\left(e^{iz}\right)^2 = \frac{-a+1}{a+1}.$$

Using that $a - 1 > 0$ and $\left(e^{iz}\right)^2 = e^{2iz}$ we rewrite this as

$$e^{2iz} = \left(\frac{a-1}{a+1}\right)(-1) = \left(\frac{a-1}{a+1}\right)e^{i\pi}.$$

Therefore

$$2iz = \ln\left(\frac{a-1}{a+1}\right) + \pi i + 2n\pi i = \ln\left(\frac{a-1}{a+1}\right) + (2n+1)\pi i.$$

Dividing by $2i$ gives

$$z = \left(n + \frac{1}{2}\right)\pi - \frac{i}{2}\ln\left(\frac{a-1}{a+1}\right)$$

for $n \in \mathbb{Z}$. □

7. *Let* $b = \dfrac{-1+i}{8}$.

(a) Find all cube roots of b *in the form* $x + iy$.
(b) Find all values of $\log(b)$ *in the form* $x + iy$.
(c) What is $\operatorname{Log}(b)$ *?*
(d) Give all values of b^{1+i} *in the form* $x + iy$.

Solution. We start by writing b in polar form in a couple of ways that will be useful to us. First note

$$|b| = \sqrt{(1/8)^2 + (1/8)^2} = \frac{\sqrt{2}}{8} = \frac{2^{1/2}}{2^3} = 2^{-5/2}.$$

$$b = 2^{-5/2} e^{\frac{3\pi}{4} i} = 2^{-5/2} e^{\frac{3\pi}{4} i + 2n\pi i}$$

(a) The cube roots of b are given by

$$\begin{aligned} b^{\frac{1}{3}} &= \left(2^{\frac{-5}{2}} e^{\frac{3\pi}{4} i + 2n\pi i}\right)^{\frac{1}{3}} \\ &= 2^{\frac{-5}{6}} e^{\frac{\pi}{4} i + \frac{2n\pi i}{3}} \end{aligned}$$

To get the thee cube roots we let $n = 0, 1, 3$ (after that the values start repeating). For $n = 0$ we have

$$\text{first cube root} = 2^{\frac{-5}{6}} e^{\frac{\pi}{4} i} = 2^{\frac{-5}{6}} \frac{1}{\sqrt{2}} (1 + i) = 2^{\frac{-4}{3}} (1 + i).$$

Letting $n = 1$ gives

$$\text{second cube root} = 2^{\frac{-5}{6}} e^{\frac{3\pi}{4} i + 2\pi i} = 2^{\frac{-5}{6}} \left(\cos\left(\frac{11\pi}{12}\right) + i \sin\left(\frac{11\pi}{12}\right)\right)$$

which does not simplify down to anything simpler. Finally letting $n = 2$

$$\text{third cube root} = 2^{\frac{-5}{6}} e^{\frac{\pi}{4} i + \frac{4\pi i}{3}} = 2^{\frac{-5}{6}} \left(\cos\left(\frac{19\pi}{12}\right) + i \sin\left(\frac{19\pi}{12}\right)\right)$$

(b) Form the definition of $\log(b)$

$$\log(b) = \ln(|b|) + i \arg(b) = \ln(2^{\frac{-5}{6}}) + i \left(\frac{3\pi}{4} + 2n\pi\right)$$

with $z \in \mathbb{Z}$.

(c) For Log(b) we choose the value of the argument between $-pi$ and pi, so

$$\text{Log}(b) = \ln(2^{\frac{-5}{6}}) + i\frac{3\pi}{4}$$

(d) Here we use the definition $b^\alpha = e^{\alpha \log b}$. We first compute $(1+i)\log(b)$:

$$\begin{aligned}(1+i)\log(b) &= (1+i)\left(\ln(2^{\frac{-5}{6}}) + i\left(\frac{3\pi}{4} + 2n\pi\right)\right) \\ &= \left(\ln(2^{\frac{-5}{6}}) - \frac{3\pi}{4} - 2n\pi\right) + \left(\ln(2^{\frac{-5}{6}}) + \frac{3\pi}{4} + 2n\pi\right) i \\ &= \alpha + \beta i\end{aligned}$$

where this defines α and β. Therefore

$$\begin{aligned}b^{1+i} &= e^{\alpha+i\beta} \\ &e^{\alpha}\left(\cos(\beta) + i\sin(\beta)\right) \\ &= e^{\ln(2^{\frac{-5}{6}})-\frac{3\pi}{4}-2n\pi}\left(\cos\left(\ln(2^{\frac{-5}{6}}) + \frac{3\pi}{4} + 2n\pi\right) + i\sin\left(\ln(2^{\frac{-5}{6}}) + \frac{3\pi}{4} + 2n\pi\right)\right) \\ &= 2^{\frac{-5}{6}}e^{-\frac{3\pi}{4}-2n\pi}\left(\cos\left(\ln(2^{\frac{-5}{6}}) + \frac{3\pi}{4} + 2n\pi\right) + i\sin\left(\ln(2^{\frac{-5}{6}}) + \frac{3\pi}{4} + 2n\pi\right)\right)\end{aligned}$$

with $n \in \mathbb{Z}$. This problem is going be be graded mostly on having the correct set up.

□

8. *Show that if α, β and γ are the interior angles of a triangle then*

$$\cos(\alpha)\cos(\beta)\cos(\gamma) - \cos(\alpha)\sin(\beta)\sin(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\sin(\beta)\cos(\gamma) = -1.$$

Hint: *This is really a corollary to Euler's Theorem $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. To see why recall that the angles of a triangle satisfy $\alpha + \beta + \gamma = \pi$ and therefore $-1 = e^{i\pi} = e^{i(\alpha+\beta+\gamma)}$.*

Solution. We have

$$\begin{aligned}
-1 = e^{i\pi} &= e^{i(\alpha+\beta+\gamma)} \\
&= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta))(\cos(\gamma) + i\sin(\gamma)) \\
&= \Big(\cos(\alpha)\cos(\beta)\cos(\gamma) - \cos(\alpha)\sin(\beta)\sin(\gamma) \\
&\qquad - \sin(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\sin(\beta)\cos(\gamma) \Big) + i(\text{stuff})
\end{aligned}$$

Taking the real part of this equation gives the result. □

We have proven the ***Cauchy-Riemann equations*** *(which we will often shorten to the "CR-equations") which are that if $f(z) = u + iv$ is analytic (that is complex differentiable) in an open U, then*

$$u_x = v_y$$
$$u_y = -v_x$$

and the derivative of f is given by either of the formulas

$$f'(z) = u_x + iv_x$$
$$f'(z) = v_y - iu_y$$

9. *The function*

$$f(z) = x^2 - y^2 + x - y + i(2xy + x + y)$$

is analytic. (This is given, so you do not have to prove it). Give a formula for the derivative $f'(z)$.

Solution. One way to do this is to use that if $f = u + iv$, then $f' = u_1 + iv_x$. In this case $u = x^2 - y^2 + x - y$ and $v = 2xy + x + y$. Thus $u_x = 2x + 1$ and $v_x = 2y + 1$ giving

$$f'(z) = 2x + 1 + i(2y + 1).$$

Anther way to do the problem is to note $f(z)$ can be rewritten as

$$f(z) = (x + iy)^2 + (1 + i)(x + iy) = z^2 + (1 + i)z.$$

When

$$f'(z) = 2z + (1 + i) = 2(x + iy) + (1 + i) = (2x + 1) + i(2y + 1).$$

□

10. *Let $f = u + iv$ be analytic on the open set U. Use the CR-equations to show the equations*

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

hold.

Solution.

$$\begin{aligned} u_{xx} + u_{yy} &= (u_x)_x + (u_y)_y \\ &= (v_y)_x + (-v_x)_y \quad \text{(by CR-equations)} \\ &= v_{yx} - v_{xy} \\ &= v_{xy} - v_{xy} \qquad \text{(as } v_{yx} = v_{xy}\text{, i.e. partial derivatives commute.)} \\ &= 0. \end{aligned}$$

Similarly

$$\begin{aligned} v_{xx} + v_{yy} &= (v_x)_x + (v_y)_y \\ &= (-u_y)_x + (u_x)_y \qquad \text{(by CR-equations)} \\ &= -u_{yx} + u_{xy} \\ &= -u_{xy} + u_{xy} \qquad \text{(partial derivatives commute.)} \\ &= 0. \end{aligned}$$

□

11. *Let h be a real valued function on the open U that satisfies*

$$h_{xx} + h_{yy} = 0$$

in U. Let $u = h_x$ and $v = -h_y$. Show that $f = u + iv$ satisfies the CR-equation.

Solution. First

$$\begin{aligned} u_x &= (h_x)_x \qquad \text{(as } u = h_x\text{)} \\ &= h_{xx} \\ &= -h_{yy} \qquad \text{(using } h_{xx} + h_{yy} = 0\text{)} \\ &= -(h_y)_y \\ &= v_y \qquad \text{(as } v = -h_y\text{)} \end{aligned}$$

Which is the first CR equation. For the second

$$\begin{aligned} u_y &= (h_x)_y && \text{(as } y = h_x\text{)} \\ &= (h_y)_x && \text{(because partial derivatives commute)} \\ &= -v_x && \text{(as } v = -h_y\text{)} \end{aligned}$$

□