Mathematics 552 Homework.

We will be studying functions of a complex variable, that is functions whose domain is a subset of the real numbers and which take values in the complex numbers. So our first step is to review some basic about complex numbers. There are several sets of numbers which I assume that you have seen before. First there is the set of ***natural numbers***

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}.$$

That is $\mathbb{N}$ is the set of positive integers. (Some people include 0 in the set of natural numbers.) The set of ***integers*** is

$$\mathbb{Z} = \{\dots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

which is the set of all positive, negative, whole numbers along with zero. If this set we can add, subtract, and multiply and get results that are still integers. But we can not divide and still get an integer. As a result it is natural extend to the ***rational numbers***

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ with } b \neq 0 \right\}.$$

In this set we can add, subtract, multiply, and divide. In particular for any linear equation

$$ax + b = 0$$

with $a, b \in \mathbb{Q}$ and $b \neq 0$ we can solve for x and get

$$x = \frac{-a}{b}.$$

But for polynomial equations with rational rational coefficients there are many equation that can not be solved inside the rational numbers. The most famous example is

$$x^2 - 2 = 0$$

which has solutions

$$x = \pm\sqrt{2}.$$

But $\sqrt{2}$ is an irrational number and therefore no solution that is a rational numbers. This is fixed by extending to the ***real numbers***

$$\mathbb{R} = \text{set of all positive and negative decimal numbers along with } 0.$$

This definition is a little vague as giving a precise definition of the real numbers is complicated and is one of the main topics covered in Math 554. In $\mathbb{R}$ there is a nice way to show that lots of equations have solutions: recall from calculus

Theorem (The Intermediate Value Theorem)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous with $f(a)$ and $f(b)$ of opposite signs (that is one is positive and one is negative). Then there is a point ξ between a and b with $f(\xi) = 0$.* □

Applying this to $f(x) = x^2 - 2$ on the integral $[1, 2]$ we have $f(1) = -1$ and $f(2) = 2$ which have opposite signs so the Intermediate Value Theorem tells us there is a ξ between 1 and 2 with $f(\xi) = \xi^2 - 2 = 0$, that is $\xi^2 = 2$. Thus $\sqrt{2}$ is a real number. See Figure 1

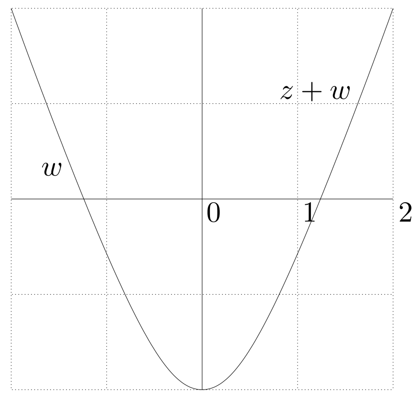

FIGURE 1. The graph of $y = x^2 - 2$ changes sign between $x = 1$ and $x = 2$. The Intermediate Value Theorem then guarantees that there is a number ξ between 1 and 2 with $\xi^2 - 2 = 0$.

Still there are some polynomial equations with real coefficients that do not have real solutions, the most natural example being

$$x^2 + 1 = 0.$$

Solving this formally leads to

$$x = \pm\sqrt{-1}$$

and -1 does not have real square root. For this equation we remedy the problem by just adding a new number to be the square root of -1 and give it a name:

$$i = \sqrt{-1}.$$

This called the ***imaginary unit*** and from its definition satisfies

$$i^2 = -1.$$

We define the ***complex numbers*** as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

and doing the algebraic operations of addition, subtraction, and multiplication in $\mathbb{C}$ follow from $i^1 = -1$ and the usual rules of algebra (commutative, associative, distribution laws). To be explicit for addition and subtraction we have

$$(a_1 + b_1) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i$$
$$(a_1 + b_1) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2)i.$$

For multiplication we have

$$\begin{aligned}(a_1+b_1i)(a_2+b_2i) &= a_1b_1 + a_1b_2i + a_2b_1i + a_2b_2i^2 && \left(\begin{matrix}\text{distributive and}\\ \text{communtative rules.}\end{matrix}\right)\\ &= (a_1b_1 - a_2b_2) + (a_1b_2 + a_2b_1)i && (\text{using } i^2 = -1).\end{aligned}$$

Division is a little more complicated.

Proposition 1. *Let $a + bi \in \mathbb{C}$ with $a + bi \neq 0$ (explicitly this means that at least one of a or b is $\neq 0$). Then $a^2 + b^2 \neq 0$ so that the complex number*

$$\frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

is defined (the denominator is not zero) and

$$(a + bi)\left(\frac{a - bi}{a^2 + b^2}\right) = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

Problem 1*.* Verify this calculation. □

This tells us how to take the multiplicative inverse of a nonzero complex number and thus how to divide. This is usual described as follows. Define the ***complex conjugate*** (or more briefly just the ***conjugate***) of $a + bi$ to be $a - bi$. Then to do the division

$$\frac{x + yi}{a + bi}$$

the trick to to multiply both the numerator and denominator by the conjugate of the denominator:

$$\frac{x + yi}{a + bi} = \frac{(x + yi)(a - bi)}{(a + bi)(a - bi)} = \frac{(ax + by) + (ay - bx)i}{a^2 + b^2} = \frac{ax + by}{a^2 + b^2} + \frac{ay - bx}{a^2 + b^2} i.$$

So in $\mathbb{C}$ we can to all four of the basic algebra operations: addition, subtraction, multiplication, and division. In the language of Math 546 the set $\mathbb{C}$ with the operations of addition and multiplication is a field.

It is traditional to denote complex numbers as

$$z = x + iy$$

where $x, y \in \mathbb{C}$. Then the ***real part*** of z is

$$\mathrm{Re}(z) = x$$

and the ***imaginary part*** of z is

$$\mathrm{Im}(z) = y.$$

Thus

$$\mathrm{Re}(3 + 4i) = 3, \qquad \mathrm{Im}(3 + 4i) = 4.$$

Note that both the real and imaginary parts are real numbers. The complex conjugate of z is denoted by $\overline{z}$. That is

$$\overline{z} = \overline{x + yi} = x - yi.$$

Proposition 2. *Let $x = x + iy$ and $w = u + iv$ be complex numbers. Then*

(a) *The conjugate of a sum is the sum of the conjugates:*

$$\overline{z + w} = \overline{z} + \overline{w}.$$

(b) *The conjugate of a product is the product of the conjugates:*

$$\overline{zw} = \overline{z}\,\overline{w}.$$

Problem 2. Prove this. □

The ***modulus***, also called the ***length***, of the complex number $z = x + iy$ is

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

Note that $|z| \geq 0$ and that $|z| = 0$ if and only if $z = 0 + 0i = 0$.

Proposition 3. *For any complex number z*

$$z\,\overline{z} = |z|^2.$$

Problem 3. Prove this. □

Proposition 4. *For any complex numbers z and w*

$$|zw| = |z||w|.$$

Problem 4. Prove this by using that $\overline{zw} = \overline{z}\,\overline{w}$ and Proposition 3. *Hint:* Start by showing $|zw|^2 = |z|^2|w|^2$. Here is part of the calculation for showing this

$$|zw|^2 = (zw)\overline{(zw)} = zw\overline{z}\,\overline{w} = z\overline{z}\,w\overline{w}.$$

But once you have $|zw|^2 = |z|^2|w|^2$ taking the positive square roots of both sides completes the proof. □

Problem 5. First some practice with doing arithmetic with complex numbers. Compute the following:

(a) $(3 - 4i)(2 + 5i)$
(b) $\dfrac{2 + 5i}{4 - 3i}$
(c) $z^2 - 2z + 2$ where $z = 1 + i$
(d) $(1 + i)^2$
(e) $(1 + i)^3$ □

Problem 6. If $z = 4 - 3i$ compute the following:

(a) $\overline{z}$
(b) $|z|$

Powers of i will come up repeatedly. The next problem shows us the pattern.

Problem 7. Compute i^2, i^3, i^4, i^5, i^6, i^7, and i^8. Then give a formula for i^n when n is a positive integer. □

Problem 8. Here is a fact that surprised me when I first learned about complex numbers, the compute number i has a complex square root. Here is how you can see this, let

$$\alpha = \frac{1+i}{\sqrt{2}}.$$

Show that $\alpha^2 = i$. $\square$.

Proposition 5. *Prove the following hold for any complex number* $z = x+yi$*.*

(a) $\operatorname{Re}(z+w) = \operatorname{Re}(z) + \operatorname{Re}(w)$*.*
(b) $\operatorname{Im}(z+w) = \operatorname{Im}(z) + \operatorname{Im}(w)$*.*
(c) $\operatorname{Re}(iz) = -\operatorname{Im}(z)$*.*
(d) $\operatorname{Im}(iz) = \operatorname{Re}(z)$*.*
(e) $|z|^2 = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2$*.*

Problem 9. Prove this. $\square$

Proposition 6. *For any complex numbers* z *and* w

(a) $z + \overline{z} = 2\operatorname{Re}(z)$*.*
(b) $z - \overline{z} = 2i\operatorname{Im}(z)$*.*
(c) $|z+w|^2 = |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2$*.*

Problem 10. Prove this. *Hint:* For (c) maybe the easiest way to get started is by writing $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w})$. $\square$

We can view a complex number $z = x + iy$ as a two dimensional vector in a obvious way: z corresponds to the point (x, y) in the plane. Then the addition of complex numbers corresponds to the vector of the vectors in the usual way. We call this the ***complex plane***.

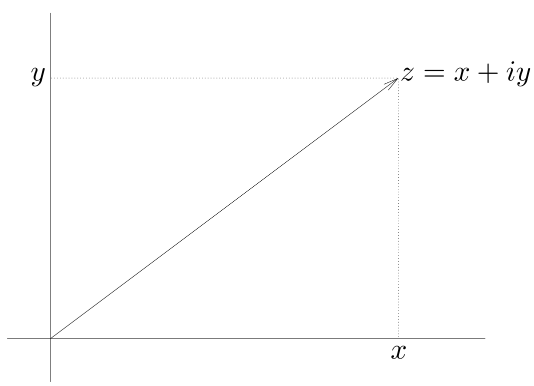

FIGURE 2. The complex number $z = x + iy$ is the point in the plane with coordinates (x, y).

We add complex numbers the same way you add vectors in the plane see Figure 3

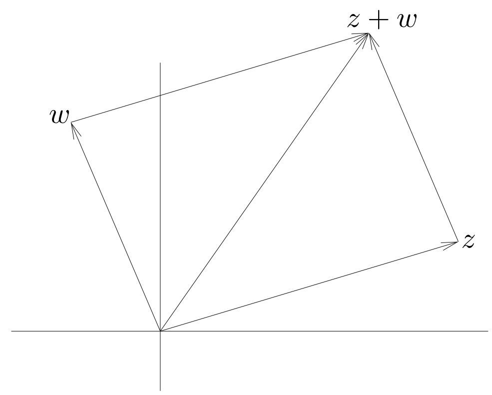

FIGURE 3. Complex numbers are added geometrically just like two dimensional vectors, by the parallelogram rule.

Example 7. Find the graph of the set of points $z = x + iy$ such that

$$\mathrm{Re}((2+3i)z) = 4.$$

Solution: We just expand the definitions involved.

$$\begin{aligned}\mathrm{Re}((2+3i)z) &= \mathrm{Re}((2+3i)(x+iy)) \\ &= \mathrm{Re}\big((2x-3y)+i(3x+2y)\big) \\ &= 2x-3y,\end{aligned}$$

Setting this equal to 4 gives $2x - 3y = 4$ which is a line in the plane. See Figure 4

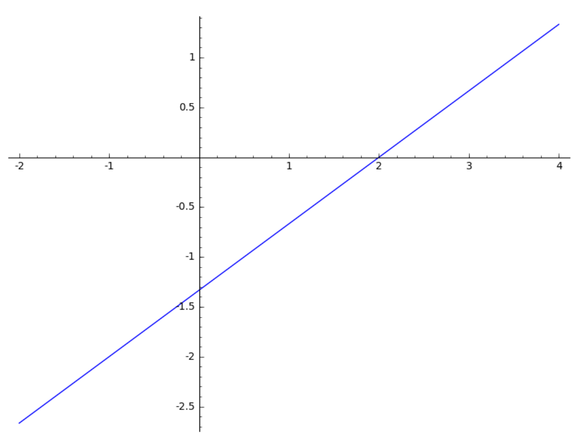

FIGURE 4. The graph of $\mathrm{Re}((2+3i)z) = 4$, which is the line with equation $2x - 3y = 4$.

□

Problem 11. Let β be the complex number $\beta = a + bi$ and assume $\beta \neq 0$. Show that if $z = x + iy$ and c is any real number then

$$\mathrm{Re}(\beta z) = c$$

is the equation of a straight line. What is the equation of this line and what is its slope? □

Recall the distance formula from vector analytic geometry with it that the distance between the points $\vec{a} = (a_1, b_1)$ and $\vec{b} = (a_2, b_2)$ is

$$\mathrm{dist}(\vec{a}, \vec{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

We can also think of this in terms of complex numbers. Let $z = x + iy$ and $w = u + iv$. Then the distance between the points z and w in the complex plane is

$$\mathrm{dist}(z, w) = \sqrt{(x - u)^2 + (y - v)^2}$$

But $z - w = (x - u) + (y - v)i$ and thus we also have $\sqrt{(x - u)^2 + (y - v)^2} = |z - w|$. Combining these gives

$$\mathrm{dist}(z, w) = |z - w|.$$

This fact is important enough to record as a proposition.

Proposition 8. *If z, w are complex numbers, then*

$$\textit{distance between } z \textit{ and } w \textit{ in the plane} = |z - w|.$$ □

Recall that a circle in the plane with center (a, b) and radius r is the set of points (x, y) that are a distance r from (a, b). That is

$$\mathrm{dist}((x, y), (a, b)) = \sqrt{(x - a)^2 + (y - b)^2} = r.$$

Writing this out in terms of complex numbers and the modulus function we have

Proposition 9. *The circle whose center is the complex number a and with radius r is the set of complex numbers z that satisfy*

$$|z - a| = r.$$ □

Thus the circle with center $a = 1 - 2i$ and radius $r = 5$ has equation

$$|z - (1 - 2i)| = 5.$$