Mathematics 552 Homework.

Theorem 1. *Let D be an open domain in $\mathbb{C}$ and $f(z)$ a function that has an anti-derivative in D. That is there is a function $F(z)$ such that*

$$F'(z) = f(z)$$

in D. Then for any curve γ in D we have the following form of the fundamental theorem of calculus:

$$\int_\gamma f(z)\,dz = F(z)\Big|_{\gamma_{initial}}^{\gamma_{end}} = F(\gamma_{end}) - F(\gamma_{initial})$$

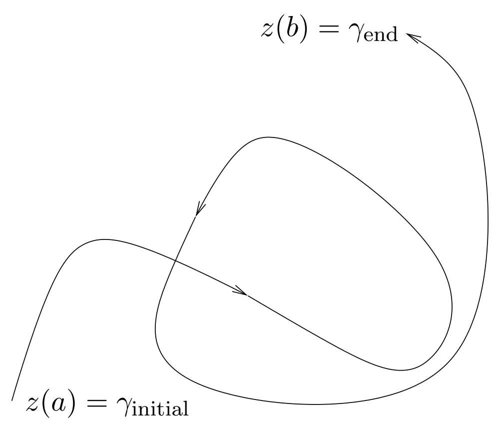
where $\gamma_{initial}$ is the initial (beginning) point of γ and γ_{end} is the end point of γ.

Proof. Let $z(t) = x(t) + iy(t)$ for $a \le t \le b$ be a parameterization of γ. Then

$$z(a) = \gamma_{\text{initial}}, \qquad z(b) = \gamma_{\text{end}}$$

and

$$dz = z'(t)\,dt$$



Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(z)z'(t)\,dt && \text{(def. of the line integral)} \\
&= \int_a^b F'(z(t))z'(t)\,dt && \text{(Using } F'(z) = f(z)\text{)} \\
&= \int_a^b \frac{d}{dt}F(z(t))\,dt && \text{(chain rule)} \\
&= F(z(b)) - F(z(a)) && \text{(math 141 version of fund. thm of calculus)} \\
&= F(\gamma_{\text{end}}) - F(\gamma_{\text{initial}})
\end{aligned}$$

□

Recall that a curve is ***closed*** if and only if $\gamma_{\text{initial}} = \gamma_{\text{end}}$. That is if and only if γ starts and ends at the same point. The following is a special case of the previous result, but is important enough to be called a theorem in its own right.

Theorem 2. *Let $f(z)$ have an antiderivative on the open domain D. Then for any closed curve γ in D*

$$\int_\gamma f(z)\,dz = 0.$$

Proof.

$$\int_\gamma f(z)\,dz = F(\gamma_{\text{end}}) - F(\gamma_{\text{initial}}) = 0.$$

□

This implies that if there is a closed curve in D such that

$$\int_\gamma f(z)\,dz \neq 0$$

then $f(z)$ does not have an antiderivative in D.

For the next couple of problems let D_a be the complex plane punctured at a. That is

$$D_a = \{z \in \mathbb{C} : z \neq a\}.$$

Problem 1. Let γ be a closed curve in D_a and let

$$f(z) = (z-a)^n$$

where n is an integer with $n \neq -1$. Show

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z-a} = 0.$$

Hint: By Theorem 1 it is enough to show that $f(z)$ has an antiderivative $F(z)$. You should be able to give an explicit formula for $F(z)$. □

Problem 2. (a) Let

$$f(z) = \frac{1}{z-a}$$

in the domain D_a. Let $r > 0$ and let γ be the circle $|z-a| = r$ traversed in the positive (that is counterclockwise) direction. Compute

$$\int_\gamma f(z)\,dz$$

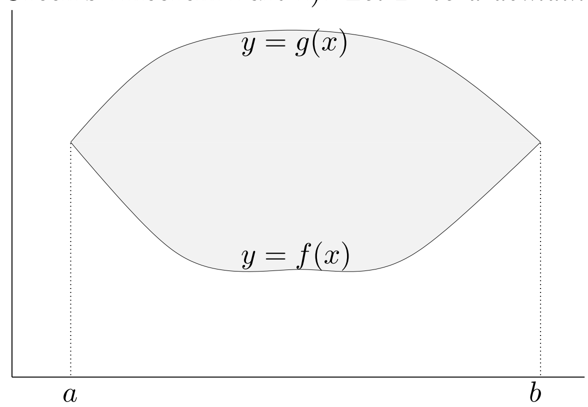
or being more explicit compute

$$\int_{|x-a|=r} \frac{dz}{z-a}.$$

Hint: The circle $|z-a| = r$ is parameterized by $z(t) = a + re^{it}$ with $0 \leq t \leq 2\pi$. With this parameterization the integral should simplify a lot.

(b) Use your answer to explain why $f(z)$ has no antiderivative in D_a. □

Theorem 3 (Green's Theorem Part 1)**.** *Let D be a domain as shown:*



Then

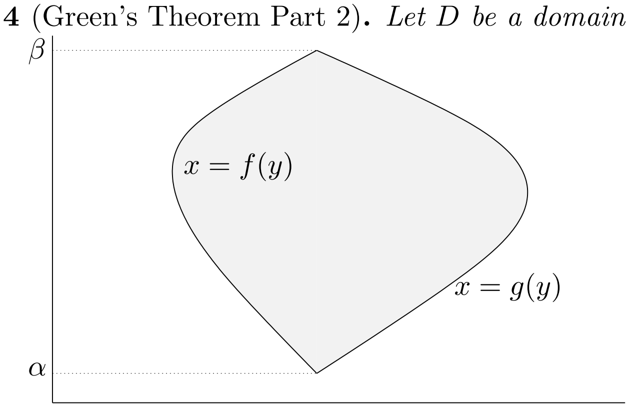
$$\int_{\partial D} P(x,y)\,dx = -\iint_D P_y(x,y)\,dx\,dx$$

Proof. Using the standard convention that we transverse the boundary keeping the inside on the left we have that

$$\begin{aligned}\int_{\partial D} P(x,y)\,dx &= \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx \\ &= -\int_a^b \left(P(x,g(x)) - P(x,f(x))\right) dx \\ &= -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx \qquad \text{(by Fundamental Theorem of Calculus)} \\ &= -\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy\end{aligned}$$

as required. □

Theorem 4 (Green's Theorem Part 2)**.** *Let D be a domain as shown:*



Then

$$\int_{\partial D} Q(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

Proof. Again orienting the direction moving around the curve so that the inside is on the left we have

$$\begin{aligned}\int_{\partial D} Q((x,y)\,dx\,dy &= \int_\alpha^\beta Q(g(y),y)\,dy - \int_\alpha^\beta Q(f(y),y)\,dy \\ &= \int_\alpha^\beta \left(Q(g(y),y) - Q(f(y),y)\right)\,dy \\ &= \int_\alpha^\beta \int_{f(y)}^{g(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy \qquad \text{(by Fundamental Theorem of Calculus)} \\ &= \iint_D \frac{\partial Q}{\partial x}(x,y)\,dx\,dy\end{aligned}$$

which is what we were to prove. □

Theorem 5 (Green's Theorem)**.** *Let D be a bounded domain with a nice boundary and let $P(x,y)$ and $Q(x,y)$ be functions that have continuous partial derivative on D and its boundary. Then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(-P_y + Q_x\right)\,dx\,dy.$$

Proof. This basically is just the two versions of Green's Theorem we have already done added together. □

Theorem 6 (Cauchy Integral Theorem)**.** *Let D be a bounded domain and $f(z) = u + iv$ be a function that satisfies the Cauchy-Riemann Equations (i.e. if $f(z)$ is analytic) on D and its boundary. Then*

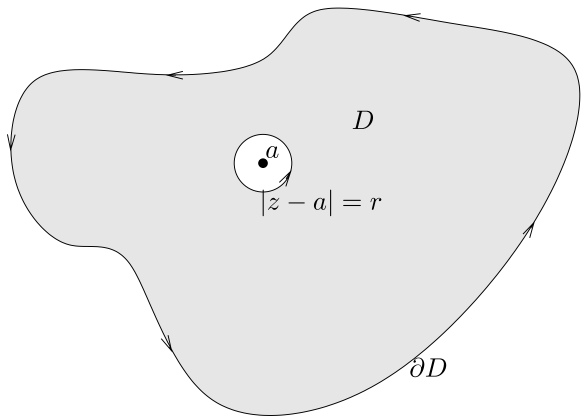
$$\int_{\partial D} f(z)\,dz = 0.$$

Proof. This is an almost straightforward application of Green's Theorem and the Cauchy-Riemann equations:

$$\begin{aligned}\int_{\partial D} f(z)\,dz &= \int_{\partial D} (u+iv)(dx+idy) \\ &= \int_{\partial D} u\,dx - v\,dy + i\int_{\partial D} v\,dx + u\,dy \\ &= \iint_D \left(-u_y - v_x\right)\,dx\,dy + i\iint_D \left(-v_y + u_x\right)\,dx\,dy \quad \text{(by Green's Theorem)} \\ &= \iint_D \left(-u_y + u_y\right)\,dx\,dy + i\iint_D \left(-u_x + u_x\right)\,dx\,dy \quad \text{(by CR equations)} \\ &= 0\end{aligned}$$

and we are done. □

Problem 3. Let $f(z)$ be analytic in a domain D and let $a \in K$. Let $r > 0$ be so small that the disk $|z-a| \le r$ is contained in D as in this figure



(a) Use the Cauchy Integral Theorem to show

$$\int_{\partial D} \frac{f(z)}{z-a}\, dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz.$$

Be sure to say why Cauchy Integral Formula applies.

(b) Use Part (a) and the parameterization of $|z-a|=r$ given by $z = a + re^{it}$ with $0 \le t \le 2\pi$ to show

$$\int_{\partial D} \frac{f(z)}{z-a}\, dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz = i \int_0^{2\pi} f(a+re^{it})\, dt.$$

Problem 4. With the same set up as in Problem 1 explain why

$$\lim_{r\to 0^+} \int_0^{2\pi} f(a+re^{it})\, dt = 2\pi f(a)$$

and use this to show

$$\int_{\partial D} \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$$

□

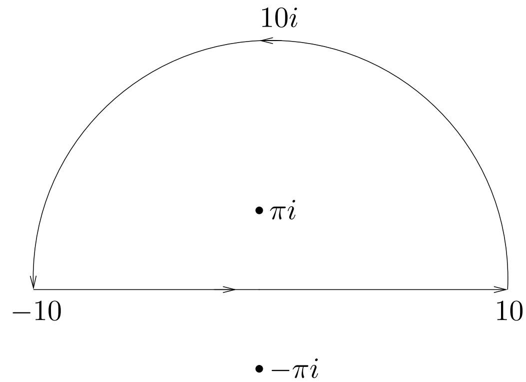
You have just proven what may be the most important result in Complex Analysis:

Theorem 7 (Cauchy Integral Formula)**.** *Let D be a bounded domain with nice boundary and $f(z)$ be analytic on D and its boundary. Then for any point $a \in D$*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\, dz}{z-a}.$$

□

Example 8. Consider the following path:



We now use the Cauchy Integral formula to evaluate

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz.$$

This function is analytic except where the denominator becomes zero. That is where $z^2+\pi^2=0$. Note that $z^2+\pi^2=(z-\pi i)(z+\pi i)$. So that the bad points are $z=\pi i$ and $z=-\pi i$. Thus our integral becomes

$$\int_\gamma \frac{e^z}{(z-\pi i)(z+\pi i)}\,dz.$$

We only need to work about the point πi as it is the only non-analytic point inside of γ. Rewrite the integral as

$$\int_\gamma \frac{e^z/(z+\pi i)}{(z-\pi i)}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz$$

where

$$f(z)=\frac{e^z}{z+\pi i}.$$

The function $f(z)$ is analytic inside of γ. So by the Cauchy integral formula

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz = 2\pi i f(\pi i) = 2\pi i\frac{e^{\pi i}}{\pi i+\pi i} = e^{\pi i} = -1.$$

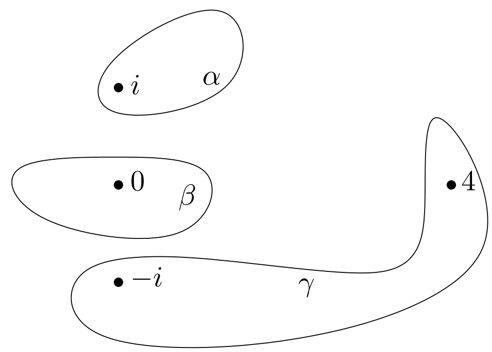
□

Problem 5. Let z_1 be a complex number and γ a simple closed curve that does not pass through z_1. Show

$$\int_\gamma \frac{dz}{z-z_1} = \begin{cases} 2\pi i, & \text{if } z_1 \text{ is inside of } \gamma, \\ 0, & \text{if } z_1 \text{ is outside of } \gamma. \end{cases}$$

Hint: Use part (d) of Problem 2, or the Cauchy Integral Formula, with $f(z)=1$, D the region inside of γ, and $z=z_1$.

Problem 6. The following figure shows the points i, $-i$, 0, and 4 along with three paths α, β, and γ.



Use the Cauchy integral formula to

(a) Evaluate $\displaystyle\int_\alpha \frac{2z+1}{z(z-4)(z^2+1)}\,dz$,

(b) Evaluate $\displaystyle\int_\beta \frac{2z+1}{z(z-4)(z^2+1)}\,dz$,

(c) Evaluate $\displaystyle\int_\gamma \frac{2z+1}{z(z-4)(z^2+1)}\,dz$.