Mathematics 552 Homework.

One of our recent results is

Theorem 1 (Basic Estimate for Complex Integrals). *Let $f(z)$ be a complex valued function defined on a curve γ. Assume there is a constant M such that*

$$|f(z)| \le M$$

for all z on the curve γ. Then

$$\left| \int_\gamma f(z)\, dz \right| \le M L(\gamma)$$

where $L(\gamma)$ is the length of γ. □

Here is an example of the use of this result. Let Assume that $|f(z)| \le 5$ on the circle $|z - 2i| = 3$. Then

$$\begin{aligned} \left| \int_{|z-2i|=3} f(z)\, dz \right| &\le 5 \times \text{Length of circle radius 3} \\ &= 5 \times 2\pi(3) \\ &= 30\pi. \end{aligned}$$

We have also shown that if $f(z)$ is analytic on the inside of a simple closed curve γ that for a inside of γ that the derivative $f'(a)$ is given by

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2}\, dz.$$

Proposition 2. *Let $f(z)$ be analytic on and inside of the circle $|z - a| = r$ and assume*

$$|f(z)| \le M \qquad \text{on the circle } |z-a| = r.$$

Then

$$|f'(a)| \le \frac{M}{r}.$$

Problem 1. Prove this along the following lines.

(a) Explain why for z on the circle $|z - a| = r$ the equality

$$\left| \frac{1}{(z-a)^2} \right| = \frac{1}{r^2}$$

holds.

(b) Let $F(z) = \dfrac{f(z)}{(z-a)^2}$ and use $|f(z)| \le M$ and part (a) to show that on the circle $|z - a| = r$ the inequality

$$|F(z)| \le \frac{M}{r^2}.$$

(c) Use Theorem 1 to explain why

$$\left| \int_{|z-a|=r} F(z)\, dz \right| \leq \frac{M}{r^2} \times 2\pi r = \frac{2\pi M}{r}.$$

(d) With this notation and the formula for $f'(a)$ we have

$$|f'(a)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2}\, dz \right| = \left| \frac{1}{2\pi i} \int_{|z-a|=r} F(z)\, dz \right|.$$

Use this to complete the proof of Proposition 2. □

Definition 3. A function $f(z)$ that is analytic on all of $\mathbb{C}$ is called an ***entire function***. □

The following is on of the more famous results in complex analysis.

Theorem 4 (Louisville's Theorem)**.** *A bounded entire function is constant.*

Problem 2*.* Prove this. *Hint:* To show a function is constant it is enough to show that its derivative is always zero. That is we want to show $f'(a) = 0$ for all a. What we know is that $f(z)$ is bounded. Explicitly this means there is a constant M so that $|f(z)| \leq M$ for all z. Let $r > 0$ and explain why

$$|f'(a)| \leq \frac{M}{r}.$$

Now take the limit as $r \to \infty$ to conclude that $|f'(a)| = 0$, and so $f'(a) = 0$ which finishes the proof. □

One of the important applications of Louisville's Theorem is showing that every polynomial with complex coefficients has a root.

We know the ***triangle inequality*** for complex numbers

$$|z + w| \leq |z| + |w|.$$

Problem 3*.* Use the triangle inequality to show for any complex numbers a, b that

$$|a + b| \geq |a| - |b|.$$

Hint: In the triangle inequality let $z = a + b$ and $w = -b$. □

Problem 4*.* Use the last problem repeatedly to show

$$|a + b_1 + b_2 + \cdots b_n| \geq |a| - |b_1| - |b_2| - \cdots - |b_n|.$$ □

Instead of working with polynomials of degree n, it will simplify notation if we work with polynomials of degree 3. All the basic ideas are the same.

Problem 5*.* Let $p(z) = z^3 + b_2 z^2 + b_1 z + b_0$. Show

$$|p(z)| \geq |z|^3 \left(1 - \frac{|b_2|}{|z|} - \frac{|b_1|}{|z|^2} - \frac{|b_0|}{|z|^3} \right).$$

Problem 6. With notation as in Problem 5 show that if $R = \max\{1, 6|b_2|, 6|b_1|, 6|b_0|\}$ then show that for $|z| \geq R$ (that is $|z| \geq 1$, $|z| \geq 6|b_2|$, $|z| \geq 6|b_1|$) that the following hold

(a) $\dfrac{1}{|z|^3} \leq \dfrac{1}{|z|^2} \leq \dfrac{1}{|z|} \leq 1$. *Hint:* This only uses $|z| \geq 1$.

(b) $\dfrac{|b_2|}{|z|} \leq \dfrac{1}{6}$. *Hint:* This uses $|z| \geq 6|b_2|$.

(c) $\dfrac{|b_1|}{|z|^2} \leq \dfrac{1}{6}$. *Hint:* This uses $|z| \geq 6|b_1|$ and part (a).

(d) $\dfrac{|b_0|}{|z|^3} \leq \dfrac{1}{6}$. *Hint:* This uses $|z| \geq 6|b_0|$ and part (a).

(e) $|p(z)| \geq \dfrac{|z|^3}{2} \geq \dfrac{1}{2}$. *Hint:* This uses parts (b), (c), (d) and Problem 4.

□

Theorem 5 (Fundamental Theorem of Algebra)**.** *Let* $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ *be a complex polynomial of degree* $n \geq 1$. *Then* $p(z)$ *has at least one complex root. That is there is at least one complex number* r *with* $p(r) = 0$.

The following problems will give a proof in the case of $n = 3$. The general case is not much harder. So we start with the polynomial

$$p(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

with $a_3 \neq 0$.

To start we note that by dividing by a_n we have that solving

$$p(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$$

is the same as solving

$$z^3 + \frac{a_2}{a_3} z^2 + \frac{a_1}{a_3} + \frac{a_0}{a_3} = 0$$

so there is no loss of generality in assuming that the lead coefficient of $p(z)$ is one. That is $p(z)$ is of the form

$$p(z) = z^3 + b_2 z^2 + b_1 z + b_0.$$

Assume, towards a contradiction, that $p(z)$ has no roots. That is $p(z) \neq 0$ for all $z \in \mathbb{C}$. Define a new function $f(z)$ by

$$f(z) = \frac{1}{p(z)}.$$

Problem 7. Explain why $f(z)$ is an entire function. That is explain why $f(z)$ is differentiable at all points. □

Problem 8. Let R be as in Problem 6. Show

$$|z| \geq R \qquad \text{implies} \qquad |f(z)| \leq 2. \qquad \square$$

Problem 9. The function $|f(z)|$ is continuous on the closed bounded set $\{z : |z| \leq R\}$, so there is a constant C such that

$$|z| \leq R \qquad \text{implies} \qquad |f(z)| \leq C.$$

(This is a basic fact from Mathematics 554, so you don't have to prove it, just copy it down to get credit.) $\square$

Problem 10. Let R be as in Problem 6 and set $M = \max\{2, C\}$. Combine Problems 8 and 9 to show

$$|f(z)| \leq M$$

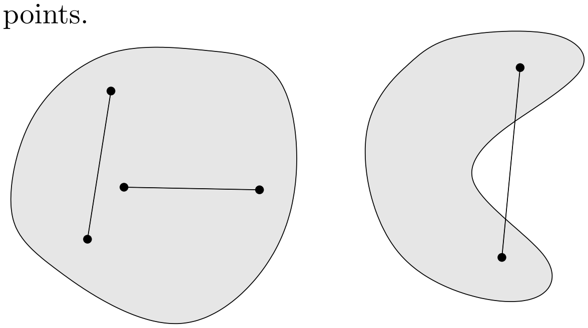
for all $z \in \mathbb{C}$. $\square$

Problem 11. Now show that $f(z) = \dfrac{1}{p(z)}$ is constant and therefore $p(z)$ is also constant. $\square$

And finally

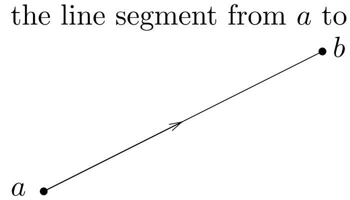
Problem 12. To finish the proof explain why the assumption $p(z)$ has no roots leads to a contradiction. *Hint:* A polynomial of degree 3 is not constant. $\square$

A set in $\mathbb{C}$ is ***convex*** if and only if it contains the line segment between any two of its points.



Convex. Not convex.

If $a, b \in \mathbb{C}$ let $\sigma_{a,b}$ be the line segment from a to b.



This is parameterized by

$$z(t) = a + t(b - a) = (1 - t)a + tb \qquad 0 \leq t \leq 1.$$

This gives

$$dz = (b - a)dt$$

and therefore if $f(z)$ is a continuous function defined on $\sigma_{a,b}$

$$(1)\quad \int_{\sigma_{a,b}} f(z)\,dz = \int_0^1 f(a+t(b-a))(b-a)\,dt = (b-a)\int_0^1 f(a+t(b-a))\,dt.$$

Note that if we reverse the order of a and b, that is use the segment $\sigma_{b,a}$ then if we do the change of variable

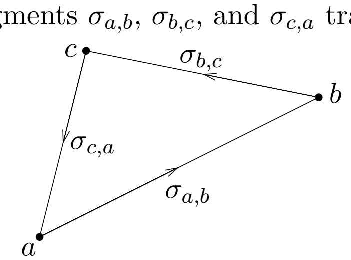
$$t = 1-s, \qquad dt = -ds$$

we get

$$\begin{aligned}\int_{\sigma_{b,a}} f(z)\,dz &= (a-b)\int_0^1 f(b+t(a-b)\,dt \\ &= (a-b)\int_1^0 f(b+(1-s)(a-b))\,(-ds) \\ &= -(b-a)\int_0^1 f(a+s(b-a))\,ds \\ &= -\int_{\sigma_{a,b}} f(z)\,dz. \qquad (2)\end{aligned}$$

That is traversing the segment in the opposite direction changes the its sign. (This is something we know form the general theory, but it is nice to see a concrete proof in a special case.)

Let $a, b, c \in \mathbb{C}$. Then the ***triangle*** with vertices a, b, and c is curve formed by the three segments $\sigma_{a,b}$, $\sigma_{b,c}$, and $\sigma_{c,a}$ traversed in that order.



Our next goal to to prove

Theorem 6 (Morera's Theorem Form 1)**.** *Let D be a convex open set in $\mathbb{C}$ and $f(z)$ a continuous function in D such that*

$$\int_T f(z)\,dz = 0$$

for every triangle in D. Then f has an antiderivative in D. That is there is a function $F(z)$ defined on D such that

$$F'(z) = f(z).$$

A restatement of the hypothesis of this theorem is that for all points $a, b, c \in D$ we have

$$\int_{\sigma_{a,b}} f(z)\,dz + \int_{\sigma_{b,c}} f(z)\,dz + \int_{\sigma_{c,a}} f(z)\,dz = 0$$

Choose a point $a \in D$ and for $b \in D$ define $F(b)$ by

$$F(b) = \int_{\sigma_{a,b}} f(z)\, dz.$$

Problem 13. Let $b \in D$ and let $h \in \mathbb{C}$ so that $b + h \in D$. Show

$$F(b+h) - F(b) = \int_{\sigma_{b,b+h}} f(z)\, dz.$$

Hint: Let T be the triagle with vertices $a, b, b + c$. Then, since we are assuming that the integral over all triangles is zero we have

$$\int_{\sigma_{a,b}} f(z)\, dz + \int_{\sigma_{b,b+h}} f(z)\, dz + \int_{\sigma_{b+h,a}} f(z)\, dz = 0 \tag{3}$$

By the definition of F we have

$$\int_{\sigma_{a,b}} f(z)\, dz = F(b).$$

And using (2) (that is reversing the direction along the segment changes the sign of the integral) we also have

$$\int_{\sigma_{b+h,a}} f(z)\, dz = -\int_{\sigma_{a,b+h}} f(z)\, dz = -F(b+h).$$

Use these equations in equation (3) to finish the problem. □

Problem 14. Use the previous problem and equation (1) to show that when $h \neq 0$

$$\frac{F(b+h) - F(b)}{h} = \int_0^1 f(b + th)\, dt.$$ □

Problem 15. Use the previous problem show for all $b \in D$ that

$$F'(b) = \lim_{h \to 0} \frac{F(b+h) - F(b)}{h} = f(b)$$

and therefore F is an antiderivative of f. This completes the proof of Theorem 6 □

Theorem 7 (Morera's Theorem Form 2)**.** *With the same hypothesis as Theorem 6 (that is that the integral of f over every triangle in D is vanishes) the function f is analytic.*

Problem 16. Prove this. *Hint:* By the first form of Morera's Theorem we know that $f(z)$ has an antiderivative F. That is $F'(z) = f(z)$. Then F analytic in D (by definition as it has a derivative). But we have shown that if a function is analytic, then so is its derivative. □