Math 554 Test 3, Answer key.

A note on notation. Let E be a metric space with metric d. Then it is wrong to write the distance between the points p and q of E as $|p-q|$ for a couple of reasons. First in a general metric space subtraction makes no sense and therefore $p-q$ is not defined. And for a metric space the absolute value $|x|$ of a point x in E is not defined. Therefore $|p-q|$ is doubly not defined. The distance between the points p and q of E is just $d(p,q)$. The only examples where $d(p,q)=|p-q|$ are when E is a subset of $\mathbb{R}$.

Recall that a function $: E \to E'$ between metric spaces is ***continuous*** if and only if it is continuous at all points of E. A good deal of our recent energy has gone into showing that this can be described in several ways:

Theorem 1. *Let $f\colon E \to E'$ be a function. Then the following are equivalent:*
(a) f is continuous.
(b) f does the right thing to convergent sequences. That is if $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} f(p_n) = f(p)$.
(c) Preimages by f of open sets are open. That is if $U \subseteq E'$ is open, then $f^{-1}[U]$ is open in E.
(d) Preimages by f of closed sets are closed. That is if $C \subseteq E'$ is closed, then $f^{-1}[C]$ is closed in E. □

Let us put this to work. To start a bit of set theory.

Proposition 2. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be functions and let $g\circ f\colon E \to E''$ be the composition $(g\circ f)(x) = g(f(x))$. Then for any subset $S \subseteq E''$*

$$(g\circ f)^{-1}[S] = f^{-1}\left[g^{-1}[S]\right]$$

Problem 1. Prove this. □

Solution. It is enough to show that

$$x \in (g\circ f)^{-1}[S] \iff x \in f^{-1}\left[g^{-1}[S]\right]$$

This is a chase through definitions.

$$\begin{aligned}
x \in (g\circ f)^{-1}[S] &\iff (g\circ f)(x) \in S && \text{(definition of preimage)}\\
&\iff g(f(x)) \in S && ((g\circ f)(x) = g(f(x)) \text{ by definition})\\
&\iff f(x) \in g^{-1}[S] && \text{(definition of preimage)}\\
&\iff x \in f^{-1}\left[g^{-1}[S]\right] && \text{(definition of preimage)}
\end{aligned}$$

which completes the proof. □

Theorem 3. *Let E, E' and E'' be metric spaces and $f\colon E \to E'$ and $g\colon E' \to E''$ continuous functions. Then the composition $g \circ f$ is also continuous.*

Problem 2. Give a short proof of this this using Condition (c) of Theorem 1 and Proposition 2. □

Solution. By Theorem 1 to show $g\circ f$ is continuous it is enough to show that for each open set $U \subseteq E''$ that $(g\circ f)^{-1}[U]$ is open set in E. So let U be an open subset of E'', then $g^{-1}[U]$ is open in E as g is continuous and the preimage of open sets by continuous functions is open. Then $f^{-1}\left[g^{-1}[U]\right]$ is open in E as f is continuous and the preimage of open sets by continuous functions is open. But by Proposition 2

$$(g \circ f)^{-1}[U] = f^{-1}\left[g^{-1}[U]\right]$$

and therefore $(g \circ f)^{-1}[U]$ is open as required. □

Back before Test 2 we proved

Theorem 4. *A metric space E and $S \subseteq E$. Then S is compact if and only if it is sequentially compact.* □

Before going on you should make sure that you know the definitions of compact (open covers have finite subcovers) and sequentially compact (sequences from the set have subsequences that converge to a point of the set).

Theorem 5. *Let $f\colon E \to E'$ be a continuous function between metric spaces and $K \subseteq E$ a compact subset of E. Then the image $f[K]$ is compact.*

Problem 3. Prove this by use of Condition (b) of Theorem 1. *Hint:* In light of Theorem 4 it is enough to show that if K is sequentially compact, so is the image $f[K]$. Let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $f[K]$. We need to find a subsequence of this sequence that converges to a point of $f[K]$. So explain why for each n that $y_n = f(x_n)$ for some $x_n \in K$. Now use the sequential compactness of K to get a subsequence $\langle x_{n_k}\rangle_{k=1}^{\infty}$ that converges to a point of K. □

Solution. Let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $f[K]$. We need to show that this sequence has a subsequence that converges to a point of $f[K]$. Because $y_n \in f[K]$ there is an $x_n \in K$ with $f(x_n) = y_n$. Because K is sequentially compact there is a subsequence $\langle x_{n_k}\rangle_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} f(x_{n_k}) = x$$

for some $x \in K$. But as f is continuous, it does the right thing to limits and thus

$$\begin{aligned}
\lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty} f(x_{n_k}) \\
&= f(x) && \text{(this is where doing right by limits is used)} \\
&\in f[K] && \text{(as } x \in K \text{ so } f(x) \in f[K])
\end{aligned}$$

Thus $\langle y_{n_k}\rangle_{k=1}^{\infty}$ is the required subsequence converging to a point of $f[K]$. □

Now some more set theory.

Proposition 6. *Let $f\colon E \to E'$ be a function between sets and $U \subseteq E'$. Then*

$$f\left[f^{-1}[U]\right] \subseteq U.$$

Remark. On the first version of the test, I messed up and had $f\left[f^{-1}[U]\right] = U$ which is only true under the extra assumption that f is surjective. □

Problem 4. Prove this. □

Solution. Let $y \in f\left[f^{-1}[U]\right]$. Then $y = f(x)$ for some $x \in f^{-1}[U]$ and by definition of the preimage $f^{-1}[U]$ this implies $f(x) \in U$. Thus $y = f(x) \in U$. As y was an arbitrary element of $f\left[f^{-1}[U]\right]$ this gives that

$$f\left[f^{-1}[U]\right] \subseteq U.$$

□

Problem 5. Give anther proof of Theorem 5 using Condition (c) of Theorem 1. *Hint:* This time we use the open cover definition of compactness. So let $\mathcal{C}$ be an open cover of $f[K]$ and we need to show it has a finite subcover. Let

$$f^{-1}[\mathcal{C}] = \{f^{-1}[U] : U \in \mathcal{C}\}.$$

Show that $f^{-1}[\mathcal{C}]$ an open cover of K and therefore $f^{-1}[\mathcal{C}]$ has a finite subcover of K as K is compact. □

Solution. Let $\mathcal{C}$ be an open cover of $f[K]$ and let $f^{-1}[\mathcal{C}]$ be as in the hint. Then, as f is continuous, Condition (c) of Theorem 1 yields that for each $U \in \mathcal{C}$ that $f^{-1}[U]$ is open. If $x \in K$, then $f(x) \in f[K]$ and thus $f(x) \in U$ for some $U \in \mathcal{C}$ as $\mathcal{C}$ is a cover of $f[K]$. Therefore $f^{-1}[\mathcal{C}]$ is an open cover of the compact set K and so there is a finite set $\{U_1, U_2, \ldots, U_m\} \subseteq \mathcal{C}$ so that

$$K \subseteq f^{-1}[U_1] \cup f^{-1}[U_2] \cup \cdots \cup f^{-1}[U_m].$$

Let $y \in f[K]$. Then $y = f(x)$ for some $x \in K$ and therefore $x \in f^{-1}[U_j]$ for some $j \in \{1, 2, \ldots, m\}$. But then $y = f(x) \in U_j$. Thus shows that $\{U_1, U_2, \ldots, U_m\}$ covers $f[K]$ as each $y \in f[K]$ is in at least one $f^{-1}[U_j]$. Thus $\mathcal{C}$ has a finite subcover of $f[K]$, showing that $f[K]$ is compact. □

Proposition 7. *Let E be a metric space and K a compact subset of E. Then K is bounded in E. More explicitly let p_0 be any point of E. Then there is a $r > 0$ so that $K \subseteq B(p_0, r)$.*

Problem 6. Prove this. *Hint:* Start by showing that

$$\mathcal{C} = \{B(p_0, n) : n = 1, 2, 3, \ldots\}$$

is an open over of K. □

Solution. Let $p \in K$. Then by Archimedes Axiom that is a natural number with $d(p_0, p) < n$. This implies $p \in B(p_0, n)$. Thus $\mathcal{C} = \{B(p_0, n) : n = 1, 2, 3, \ldots\}$ is an open cover of K. As K is compact this implies there is a finite subcover, that is there are positive integers $n_1, n_2, \ldots, n_m$ with

$$K \subseteq B(p_0, n_1) \cup B(p_0, n_2) \cup \cdots \cup B(p_0, n_m) = B(p_0, r)$$

where

$$r = \max\{n_1, n_2, \ldots, n_m\}$$

which finishes the proof. □

The next result is generally considered on of the big theorems for this course.

Theorem 8. *Let E be a compact metric space and $f \colon E \to \mathbb{R}$ a continuous function. Then f is bounded on E. Explicitly this means that there is a positive number r so that $-r \leq f(p) \leq r$ for all $p \in E$.*

Problem 7. Prove this. *Hint:* Explain by the set $f[E]$ is compact in $\mathbb{R}$. Now Proposition 7 may be relevant. □

Solution. Recall that in $\mathbb{R}$ the open balls are intervals:

$$B(y, r) = (y - r, y + r).$$

By Theorem 5 the set $f[E]$ is a compact subset of $\mathbb{R}$. Now in Proposition 7 use $p_0 = 0 \in \mathbb{R}$ to find $r > 0$ so that

$$f[K] \subseteq B(0, r) = (0 - r, 0 + r) = (-r, r).$$

Then if $p \in K$ we have $f(p) \in (-r, r)$, that is $-r < f(p) < r$ for all $p \in K$. □

Problem 8. Give examples of the following:

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at all point other than -4 and 7 but discontinuous at -4 and 7.
(b) A closed subset of $\mathbb{R}$ that is not compact.
(c) A bounded subset of $\mathbb{R}$ that is not compact.
(d) A subset of $\mathbb{R}$ that is neither open or closed.

Solution. (a) An example that does not quite work is

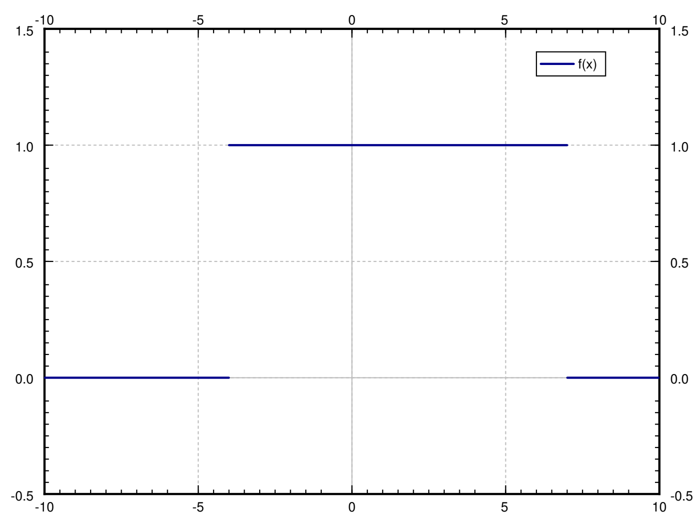
$$f(x) = \frac{1}{(x+4)(x-7)}.$$

Why it only almost works is that it is not defined on all of $\mathbb{R}$. We can fix this by defining $f(x)$ to be any values we please for $x = -4$ and 7 for example

$$f(x) = \begin{cases} \dfrac{1}{(x+4)(x-7)}, & x \neq -4, 7; \\ 42, & x = -4; \\ 17, & x = 7. \end{cases}$$

Anther easy example is

$$f(x) = \begin{cases} 1, & -4 < x < 7; \\ 0, & \text{otherwise}. \end{cases}$$



(b) Examples are $\mathbb{R}$, $[0, \infty)$ and the set of integer $\mathbb{Z}$. Any closed set that is not bounded works.
(c) Maybe the most natural example is the open interval $(0, 1)$. Anther example is the set $\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$. An bounded set that is not closed works.

(d) A half open interval such as $[0, 1)$ works. Anther nice example is the set, $\mathbb{Q}$, of rational numbers. □

□

Our most basic form of the Intermediate Value Theorem is

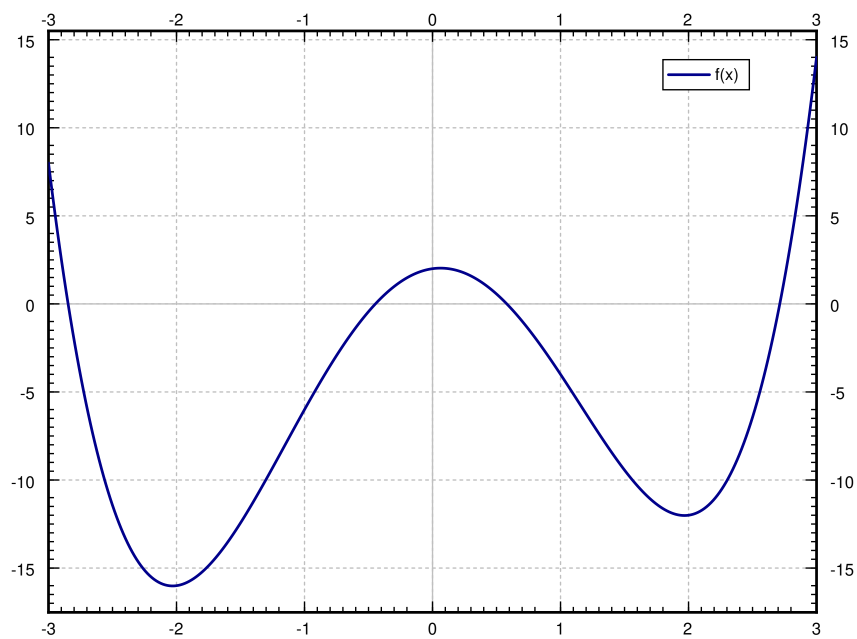
Theorem 9. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous and assume that $f(a)$ and $f(b)$ have opposite signs (that is either $f(a) > 0$ and $f(b) < 0$ or $f(a) < 0$ and $f(b) > 0$). Then $f(x) = 0$ has a solution for some x between a and b.* □

Problem 9. Use the Intermediate Value Theorem to prove the equation

$$x^4 - 8x^2 + x + 2 = 0$$

has four solutions. *Hint:* I would start by graphing it to see if this is reasonable. □

Solution. Plotting $f(x) = x^4 - 8x^2 + x + 2$ between $x = -3$ and $x = 3$ gives



which gives an idea where the roots are located. We now compute some values of $f(x)$.

$$\begin{aligned} f(-3) &= 8 \\ f(-1) &= -6 \\ f(0) &= 2 \\ f(1) &= -4 \\ f(3) &= 14 \end{aligned}$$

Thus $f(x)$ changes sign on $[-3,-1]$ and therefore $f(x) = 0$ has a solution between -3 and -1.

$f(x)$ changes sign on $[-1,0]$ and therefore $f(x) = 0$ has a solution between -1 and 0.

$f(x)$ changes sign on $[0,1]$ and therefore $f(x) = 0$ has a solution between 0 and 1.

Finally $f(x)$ changes sign between 1 and 3 and therefore $f(x) = 0$ has a solution between 1 and 3.

This gives use four solutions. (And there can not be any more than four as a degree four polynomial has at most four solutions.)

$\square$