Mathematics 554 Homework.

Problem 1. Prove the following generalization of Rollê's rule. Let f' and f'' exist on an interval (a,b) and assume there are $x_0, x_1, x_2 \in (a,b)$ with $x_0 < x_1 < x_2$ and

$$f(x_0) = f(x_1) = f(x_2) = 0.$$

There there is a point ξ between x_0 and x_2 with $f''(\xi) = 0$. □

Problem 2. Let $f\colon (a,b) \to \mathbb{R}$ so that f' and f'' exist on (a,b). Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial. Assume there are distinct points $x_0, x_1, x_2 \in (a,b)$ with $x_0 < x_1 < x_2$ and

$$f(x_0) = p(x_0), \quad f(x_1) = p(x_1), \quad f(x_2) = p(x_2).$$

There there is a point ξ between x_0 and x_2 with

$$f''(\xi) = 2a.$$

□

Remark 1. The previous problem can be viewed as a generalization of the mean value theorem. One way to think of the mean value theorem is that if f is a differentiable function and $p(x) = mx+b$ is linear and $f(x_0) = p(x_0)$ and $f(x_1) = p(x_1)$ for distinct points x_0 and x_1 then there is a point ξ between x_0 and x_1 with $f'(\xi) = m$. But m is the slope of the line and is given by $m = (f(x_1) - f(x_0))/(x_1 - x_0)$. In the quadratic case if $p(x) = ax^2 + bx + c$ is a quadratic what agrees with f at the points x_0, x_1, and x_2, then the lead coefficient a of p is

$$a = \frac{1}{(x_2 - x_0)} \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right)$$

which is a "second order" slope corresponding to the fact that $p(x)$ has degree 2. There are higher order versions of this where for a function that agrees with a degree n polynomial at $n+1$ points. □

Problem 3. Let $f\colon (a,b) \to \mathbb{R}$ be so that for some $x_0 \in (a,b)$ that $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$. Show $f(x) \geq f(x_0)$ for all $x \in (a,b)$. That is x_0 is a global minimizer of f on (a,b). □

Problem 4. In the notes do Problems 2.21(b)(c), 2.29(a), 2.32.