Mathematics 554 Homework.

On Test 1 you have have to give a proof of either Theorem 1 or Theorem 2 below. So part of this homework is making sure you understand the proofs and can reproduce them.

Let us review some of what we have covered recently. Let C be the graph of $y = f(x)$ with $a \le x \le b$. Then we can parameterize this by

$$\mathbf{r}(x) = (x, f(x))$$

or in parametric form

$$x = x \qquad dx = dx$$
$$y = f(x) \qquad dy = f'(x)\,dx.$$

so that

$$\int_C P(x,y)\,dx + Q(x,y)\,dx = \int_a^b \big(P(x,f(x)) + Q(x,f(x))f'(x)\big)\,dx$$

If $Q \equiv 0$ this simplifies to

$$\int_C P(x,y)\,dx = \int_a^b P(x,f(x))\,dx$$

We also want to recall the Fundemential Theorem of Calculus. In its most basic form it is

$$\int_a^b F'(x)\,dx = F(x)\Big|_{x=a}^{b} = F(b) - F(a).$$

Since it does not matter what variable we use in an integral we also have

$$\int_a^b F'(y)\,dy = F(y)\Big|_{y=a}^{b} = F(b) - F(a).$$

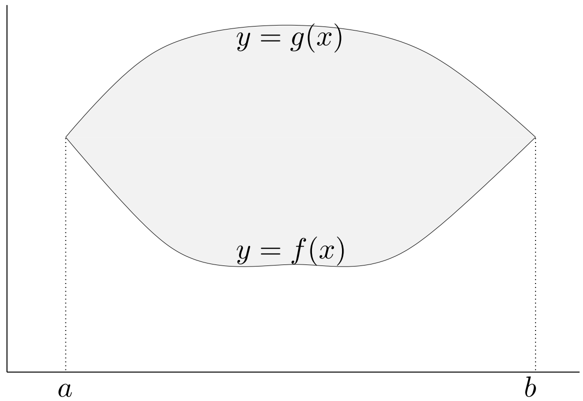
This can be generalized to

$$\int_a^b \frac{\partial P}{\partial y}(x,y)\,dy = P(x,b) - P(x,a).$$

Or more generally yet we can let a and b depend on x, for as far as integration with respect to y is concerned x is a constant and thus so are $f(x)$ and $g(x)$. Thus the Fundemential Theorem of Calculus in this case gives us that

$$\int_{g(x)}^{f(x)} \frac{\partial P}{\partial y}(x,y)\,dy = P(x,f(x)) - P(x,g(x)).$$

Theorem 1 (Green's Theorem Part 1)**.** *Let D be a domain as shown:*



Then

$$\oint_{\partial D} P(x,y)\,dx = -\iint_D P_y(x,y)\,dA$$

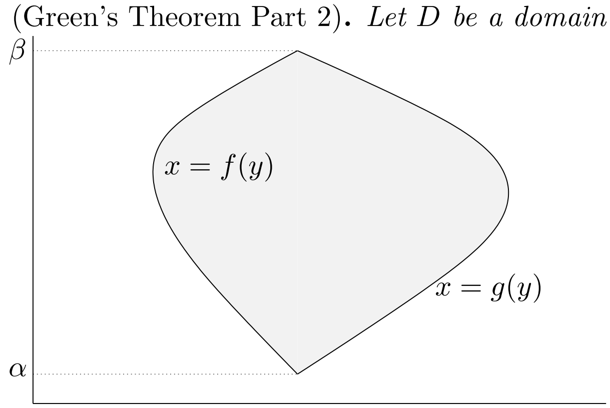
where $dA = dx\,dy = dy\,dx$ *is integration with respect to area.*

Proof. Using the standard convention that we transverse the boundary keeping the inside on the left we have that

$$\begin{aligned}
\oint_{\partial D} P(x,y)\,dx &= \int_a^b P(x,f(x))\,dx - \int_a^b P(x,g(x))\,dx \\
&= -\int_a^b \left(P(x,g(x)) - P(x,f(x))\right)dx \\
&= -\int_a^b \int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx \qquad \text{(by Fundamental Theorem of Calculus)} \\
&= -\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy
\end{aligned}$$

as required. □

Theorem 2 (Green's Theorem Part 2)**.** *Let* D *be a domain as shown:*



Then

$$\oint_{\partial D} Q(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}(x,y)\,dA$$

Proof. Again orienting the direction moving around the curve so that the inside is on the left we have

$$\begin{aligned}\oint_{\partial D} Q(x,y)\,dx\,dy &= \int_\alpha^\beta Q(g(y),y)\,dy - \int_\alpha^\beta Q(f(y),y)\,dy \\ &= \int_\alpha^\beta \big(Q(g(y),y) - Q(f(y),y)\big)\,dy \\ &= \int_\alpha^\beta \int_{f(y)}^{g(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy \qquad \text{(by Fundamental Theorem of Calculus)} \\ &= \iint_D \frac{\partial Q}{\partial x}(x,y)\,dx\,dy\end{aligned}$$

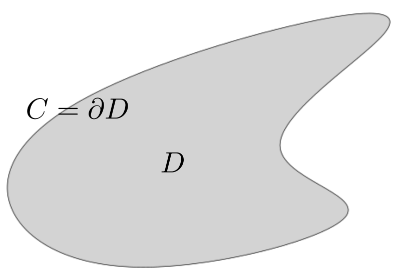
which is what we were to prove. □

Theorem 3 (Green's Theorem)**.** *Let D be a bounded domain with a nice boundary and let $P(x,y)$ and $Q(x,y)$ be functions that have continuous partial derivative on D and its boundary. Then*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \big(-P_y + Q_x\big)\,dx\,dy.$$

Proof. This basically is just the two versions of Green's Theorem we have already done added together. □

Problem 1*.* Let C be a simple closed curve as pictured and let D be the region inclosed by D.



We transverse the curve keeping the interion on the left (that is counterclockwise). Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(D).$$

Problem 2*.* Find the area inclosed by the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$. *Hint:* Start by showing this ellipse is parameterized by $x = a\cos(t)$, $y = b\sin(t)$.

Problem 3*.* For practice in using Green's Theorem in *Vector Calculus*on page 155 do problems 1–4.

Problem 4. We have done this in class, but it is worth doing again to make sure we understand it. Let C be a simple closed curve in the place and D the region it inclosed. Let $\mathbf{r}\colon [a,b] \to \mathbb{R}^2$ be a parameterization of C which transverses C so that the inside of D is on the left. Then we have seen that the outward pointing unit normal to D along C is

$$\mathbf{n}(t) = \|\mathbf{r}'(t)\|^{-1}(y'(t)\mathbf{i} - x'(t)\mathbf{j})$$

where $\mathbf{r}(x) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Let

$$\mathbf{F} = P\mathbf{i} + q\mathbf{j}$$

be a vector field. Let $ds = \|\mathbf{r}'(t)\|\, dt$ be arclength measure along C.

(a) Show

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b -Q\, dx + P\, dy.$$

(b) Apply Green's Theorem to this to get

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div}(\mathbf{F})\, dA$$

where $\operatorname{div}(\mathbf{F}) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ is the ***divergence*** of $\mathbf{F}$. This is the Divergence Theorem.

One form of the chain rule for functions $f = f(x,y)$ of two variables is

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Definition 4. Let $f = f(x,y)$ be a differentiable function of two variables. Define the ***gradient*** of f to be the vector field

$$\nabla f := \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Problem 5. Compute the gradient of the following

(a) $f(x,y) = \dfrac{1}{2}(x^2 + y^2)$.

(b) $g(x,y) = x^2 e^{x,y}$

(c) $h(x,y) = \ln(x^2 + y^2)$.

Problem 6. Let $\mathbf{V}$ be a vector field and f a scalar function. Show

$$\operatorname{div}(f\mathbf{V}) = \nabla f \cdot \mathbf{V} + f \operatorname{div}(\mathbf{V}).$$

Problem 7. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ and $f = f(x,y)$ a function of two variables. Show the chain rule can be rewritten as

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Problem 8. Let $\mathbf{r}\colon [a,b] \to \mathbb{R}^2$ be a parameterization of the curve C and $f(x,y)$ a differentiable function.

(a) Chase through definitions to show

$$\int_C \nabla f \bullet d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt.$$

(b) Use this and the Fundamental Theorem of Calculus to show

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(t))\Big|_{t=a}^{b} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$